STAT 804 Solutions

Assignment 1

1. Let ϵ_t be a Gaussian white noise process. Define

$$X_t = \epsilon_{t-2} + 4\epsilon_{t-1} + 6\epsilon_t + 4\epsilon_{t+1} + \epsilon_{t+2}.$$

Compute and plot the autocovariance function of X.

Solution:

$$R_X(h) = \begin{cases} 70\sigma^2, & h = 0 \\ 56\sigma^2, & h = 1 \\ 28\sigma^2, & h = 2 \\ 8\sigma^2, & h = 3 \\ \sigma^2, & h = 4 \\ 0, & h \geq 5 \end{cases}$$

To plot this you can pick a value for σ, say 1 or 1/70 which would give the autocorrelation.

2. Suppose that X_t is strictly stationary.

   (a) If g is some function from R^{p+1} to R show that

$$Y_t = g(X_t, X_{t-1}, \ldots, X_{t-p})$$

is strictly stationary.

Solution: You must prove the following assertion: for any k and any $A \subset R^k$ we have

$$P((Y_{t+1}, \ldots, Y_{t+k}) \in A) = P((Y_1, \ldots, Y_k) \in A)$$

(for the mathematically inclined you need this for "Borel sets A".) Define g^* by

$$g^*(x_{1-p}, \ldots, x_k) = (g(x_1, x_0, \ldots, x_{1-k}), \ldots, g(x_k, \ldots, x_{k-p}))$$

so that

$$(Y_{t+1}, \ldots, Y_{t+k}) = g^*(X_{t+1-p}, \ldots, X_{t+k})$$

and

$$(Y_1, \ldots, Y_k) = g^*(X_{1-p}, \ldots, X_k)$$

Then

$$P((Y_{t+1}, \ldots, Y_{t+k}) \in A) = P((X_{t+1-p}, \ldots, X_{t+k}) \in B)$$

where

$$B = (g^*)^{-1}(A)$$

is the inverse image of A under the map g^*. In fact the probability on the right is the definition of the probability on the left!

(REMARK: Students sometimes worry about whether or not you could take this $(g^*)^{-1}(A)$; I suspect they are worried about the existence of a so-called functional inverse of g^*. The latter exists only if g^* is a bijection: one-to-one and onto. But the inverse image B of A exists for any g^*; it is defined as $\{x : g^*(x) \in A\}$. As a simple example if $g^*(x) = x^2$ then there is no functional inverse of g^* but for instance,

$$(g^*)^{-1}([1,4]) = \{x : 1 \le x^2 \le 4\} = [-2,-1] \cup [1,2]$$

so that the inverse image of $[1,4]$ is perfectly well defined.)

For the special case $t = 0$ we also get

$$P((Y_1, \ldots, Y_k) \in A) = P((X_{1-p}, \ldots, X_k) \in B)$$

But since X is stationary

$$P((X_{t+1-p}, \ldots, X_{t+k}) \in B) = P((X_{1-p}, \ldots, X_k) \in B)$$

from which we get the desired result.

(b) What property must g have to guarantee the analogous result with strictly stationary replaced by 2^{nd} order stationary? [Note: I expect a sufficient condition on g; you need not try to prove the condition is necessary.]

Solution: If g is affine, that is $g(x_1, \ldots, x_p) = Ax + b$ for some $1 \times p$ vector A and a constant b then Y will have stationary mean and covariance if X does. In fact I think the condition is necessary but do not know a complete proof. In your solutions I wanted to see a computation of the autocovariance of the new process from that of the old.

3. Suppose that ϵ_t are iid and have mean 0 with finite variance. Verify that $X_t = \epsilon_t \epsilon_{t-1}$ is stationary and that it is wide sense white noise.

Solution: We have $\mathrm{E}(X_t) = \mathrm{E}(\epsilon_t)\mathrm{E}(\epsilon_{t-1}) = 0$. The autocovariance function of X is

$$R_X(h) = \begin{cases} \mathrm{E}(\epsilon_t^2)\mathrm{E}(\epsilon_{t-1}^2) & h = 0 \\ \mathrm{E}(\epsilon_{t+1})\mathrm{E}(\epsilon_t^2)\mathrm{E}(\epsilon_{t-1}) - \mu^4 & h = 1 \\ \mathrm{E}(\epsilon_{t+h})\mathrm{E}(\epsilon_{t+h-1})\mathrm{E}(\epsilon_t)\mathrm{E}(\epsilon_{t-1}) & h \ge 2 \end{cases} = \begin{cases} \sigma^4 \\ 0 \\ 0 \end{cases}$$

Thus X_t is second order white noise. In fact, from question 2 this sequence is strongly stationary. It is also second order white noise but it is not strict sense white noise.

4. Suppose X_t is a stationary Gaussian series with mean μ_X and autocovariance $R_X(k)$, $k = 0, \pm 1, \ldots$. Show that $Y_t = \exp(X_t)$ is stationary and find its mean and autocovariance.

Solution: The stationarity comes from question 2. To compute the mean and covariance of Y we use the fact that the moment generating function of a $N(\mu, \sigma^2)$ random variable is $\exp(\mu s + \sigma^2 s^2/2)$. Since $E(Y_t)$ is just the mgf of X_t at $s = 1$ we see that the mean of Y is just $\exp(\mu_X + R_X(0)/2)$. To compute the covariance we need

$$E(Y_t Y_{t+h}) = E(\exp(X_t + X_{t+h}))$$

which is just the mgf of $X_t + X_{t+h}$ at 1. Since $X_t + X_{t+h}$ is $N(2\mu_x, 2R_X(0) + 2R_X(h))$ we see that the autocovariance of Y is

$$C_Y(h) = \exp(2\mu_X + R_X(0) + R_X(h)) - \exp(2(\mu_X + R_X(0)/2))$$

or

$$C_Y(h) = \exp(2\mu_X + R_X(0))(\exp(R_X(h)) - 1)$$

5. Suppose that

$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + \epsilon_t$$

where ϵ_t is an iid mean 0 sequence with variance σ_ϵ^2. Compute the autocovariance function and plot the results for $\rho_1 = 0.2$ and $\rho_2 = 0.1$. (NOTE: I mean ρ_i and NOT a_i here.) I have shown in class that the roots of a certain polynomial must have modulus more than 1 for there to be a stationary solution X for this difference equation. Translate the conditions on the roots $1/\alpha_1, 1/\alpha_2$ to get conditions on the coefficients a_1, a_2 and plot in the a_1, a_2 plane the region for which this process can be rewritten as a causal filter applied to the noise process ϵ_t.

Solution: This is my rephrasing of the question. To compute the autocovariance function you have two possibilities. First you can factor

$$(I - a_1 B - a_2 B^2) = (I - \alpha_1 B)(I - \alpha_2 B)$$

with the $1/\alpha_i$ the roots of $1 - a_1 x - a_2 x^2 = 0$ and then write, as in class,

$$X_t = \sum_{k=0}^{\infty} b_k \epsilon_{t-k}$$

where

$$b_k = \sum_{\ell} = 0^k \alpha_1^\ell \alpha_2^{k-\ell}$$

The autocovariance function is then

$$C_X(k) = \sigma_\epsilon^2 \sum_{j=0}^{\infty} b_j b_{j+k}$$

This would be rather tedious to compute; you would have to decide how many terms to take in the infinite sums.

The second possibility is the recursive method:

$$C_X(h) = \mathrm{Cov}(X_{t+h}, X_t) = a_1 C_X(h-1) + a_2 C_X(h-2)$$

To get started you need values for $C_X(0)$ and $C_X(1)$. The simplest thing to do, since the value of σ_ϵ^2 is free to choose when you plot, is to just assume $C_X(0) = 1$ so that you just compute the autocorrelation function. To get $C_X(1)$ put $h = 1$ in the recursion above and get

$$C_X(1) = a_1 + a_2 C_X(1)$$

so that $\rho_X(1) = a_1/(1-a_2)$. Divide the recursion by $C_X(0)$ to see that the recursion is then

$$\rho_X(h) = a_1\rho_X(h-1) + a_2\rho_X(h-2)\,.$$

You can use this for $h \geq 2$.

Now the roots $1/\alpha_i$ are of the form

$$\frac{a_1 \pm \sqrt{a_1^2+4a_2}}{-2a_2}$$

The stationarity conditions are that both of these roots must be larger than 1 in modulus.

If $a_1^2 + 4a_2 \geq 0$ then the two roots are real. Set them equal to 1 and then to -1 to get the boundary of the region of interest:

$$a_1 \pm \sqrt{a_1^2+4a_2} = -2a_2$$

gives $a_1^2 + 4a_2^2 + 4a_1a_2 = a_1^2 + 4a_2$ or, for $4a_2 \neq 0$ we get $a_1 + a_2 = 1$. Similarly, setting the root equal to -1 gives

$$a_2 - a_1 = 1$$

It is now not hard to check that the inequalities

$$a_1 + a_2 < 1$$

$$a_2 - a_1 < 1$$

and

$$a_1^2 + 4a_2 \geq 0$$

guarantee, for $a_2 \neq 0$ that the roots have absolute value more than 1.

When the discriminant $a_1^2 + 4a_2$ is negative the two roots are complex conjugates

$$\frac{a_1}{-2a_2} \pm i\frac{\sqrt{-4a_2 - a_1^2}}{-2a_2}$$

and have modulus squared

$$1/|a_2|$$

which will be more than 1 provided $|a_2| < 1$.

Finally for $a_2 = 0$ the process is simply an AR(1) which will be stationary for $|a_1| < 1$. Putting together all these limits gives a triangle in the a_1, a_2 plane bounded by the lines $a_1 + a_2 = 1$, $a_2 - a_1 = 1$ and $a_2 = -1$. (The boundaries of the triangle are not included in the set corresponding to a stationary process.)

6. Suppose that ϵ_t is an iid mean 0 variance σ_ϵ^2 sequence and that $a_t; t = 0, \pm1, \pm2, \ldots$ are constants. Define

$$X_t = \sum a_s \epsilon_{t-s}.$$

(a) Derive the autocovariance of the process X.

(b) Show that $\sum a_s^2 < \infty$ implies

$$\lim_{N\to\infty} E[(X_t - \sum_{-N}^{N} a_s \epsilon_{t-s})^2] = 0$$

This condition shows that the infinite sum defining X converges "in the sense of mean square". It is possible to prove that this means that X can be defined properly. [Note: I don't expect much rigour in this calculation. Mathematically, you can't just define X_t as this question supposes since the sum is infinite. A rigourous treatment — WHICH I DO NOT EXPECT — asks you to prove that the condition $\sum a_s^2 < \infty$ implies that the sequence $S_N \equiv \sum_{-N}^{N} a_s \epsilon_{t-s}$ is a Cauchy sequence in L^2. Then you have to know that this implies the existence of a limit in L^2 (technically, the point is that L^2 is a Banach space). Then you have to prove that the calculation you made in the first part of the question is mathematically justified.]

Solution to a:

$$C_X(h) = \mathrm{Cov}(\sum a_s \epsilon_{t+h-s}, \sum_u a_u \epsilon_{t-u})$$

simplifies to

$$C_X(h) = \sum_s a_s a_{s-h} \, .$$

Solution to b: I had in mind the simple calculation

$$X_t - \sum_{-N}^{N} a_s \epsilon_{t-s} = \sum_{|s|>N} a_s \epsilon_{t-s}$$

which has mean 0 and variance

$$\sum_{|s|>N} a_s^2$$

The latter quantity converges to 0 since

$$\sum_{|s|>N} a_s^2 = \sum_s a_s^2 - \sum_{-N}^{N} a_s^2 \to 0$$

More rigour requires the following ideas. I had no intention for students to discover or use these ideas but some, at least, were interested to know.

Let L_2 be the set of all random variables X such that $E(X^2) < \infty$ where we agree to regard two random variables X_1 and X_2 as being the same if $E((X_1 - X_2)^2) = 0$. (Literally we define them to be equivalent in this case and then let L_2 be the set of equivalence classes.) It is a mathematical fact about L_2 that it is a Banach space, or a complete normed vector space with a norm defined by $||X|| = \sqrt{E(X^2)}$. The important point is that any Cauchy sequence in L_2 converges to some limit.

Define $S_N = \sum_{-N}^{N} a_s \epsilon_{t-s}$ and note that for $N_1 < N_2$ we have

$$||S_{N_2} - S_{N_1}||^2 = \sum_{N_1 < n \le N_2} a_s^2 \le \sum_{n > N_1} a_s^2$$

which shows that S_N is Cauchy because the sum converges. Thus there is an S_∞ such that $S_N \to S_\infty$ in L_2 which means

$$E((S_\infty - S_N)^2) \to 0$$

This S_∞ is precisely our definition of X_t.

7. Given a stationary mean 0 series X_t with autocorrelation ρ_k, $k = 0, \pm 1, \ldots$ and a fixed lag D find the value of A which minimizes the mean squared error

$$E[(X_{t+d} - AX_t)^2]$$

and for the minimizing A evaluate the mean squared error in terms of the autocorrelation and the variance of X_t.

Solution: You get

$$\begin{aligned} E[(X_{t+d} - AX_t)^2] &= E(X_{t+d}^2) - 2AE(X_{t+d}X_t) + A^2 E(X_t^2) \\ &= C_X(0) - 2AC_X(d) + A^2 C_X(0) \end{aligned}$$

this quadratic is minimized when its derivative $-2C_X(d) + 2AC_X(0)$ is 0 which is when

$$A = C_X(d)/C_X(0) = \rho_d$$

Put in this value for A to get a mean squared error of

$$C_X(0) - 2\rho_d C_X(d) + \rho_d^2 C_X(0) = C_X(0)(1 - 2\rho_d^2 + \rho_d^2)$$

or just

$$C_X(0)(1 - \rho_d^2)\,.$$

8. The semivariogram of a stationary process X is

$$\gamma_X(m) = \frac{1}{2} E[(X_{t+m} - X_t)^2]\,.$$

(Without the 1/2 it's called the variogram.) Evaluate γ in terms of the autocovariance of X.

Solution:

$$\begin{aligned} \gamma_X(m) &= \frac{1}{2} E[(X_{t+m} - X_t)^2] \\ &= \frac{1}{2} E[(X_{t+m} - \mu)^2 + (X_t - \mu)^2 - 2(X_{t+m} - \mu)(X_t - \mu)] \\ &= \frac{1}{2}(C_X(0) + C_X(0) - 2C_X(m)) \\ &= C_X(0)(1 - \rho_X(m))\,. \end{aligned}$$

9. A process X_t follows a an ARCH(1) model if the conditional distribution of X_{t+1} given $X_t, X_{t-1}, \cdots$ is normal with mean 0 and variance $a + bX_t^2$. Develop conditions on the parameters a and b for there to exist a stationary series with this behaviour. (There is an analogy to what I did for AR(1) processes.) Hint: Let $Z_{t+1} = X_{t+1}/(a + bX_t^2)^{1/2}$. What is the conditional distribution of Z_{t+1} given all previous Xs? What is the joint distribution of all the Zs? Can you define the Xs from the Zs?

   Solution: If either a or b is negative then there is a positive probability that $a+bX_t^2 < 0$ so we must have $a \geq 0$ and $b \geq 0$. If X_t has finite variance then

$$\mathrm{E}(X_{t+1}) = \mathrm{E}\left[\mathrm{E}\left(X_{t+1}|X_t\right)\right] = 0$$

   and

$$\begin{aligned}\mathrm{E}(X_{t+1}^2) =& \mathrm{E}\left[\mathrm{E}\left(X_{t+1}^2|X_t\right)\right] \\ =& \mathrm{E}\left[a + bX_t^2\right]\end{aligned}$$

   If $\sigma_X^2 = \mathrm{E}(X_t^2)$ then we get

$$\sigma_X^2 = \frac{a}{1-b}$$

   from which we deduce that $a > 0$ (unless X is constant) and $b < 1$.

   On the other hand if $a > 0$ and $0 \leq b < 1$ then we may wreite

$$\begin{aligned}X_{t+1} =& Z_{t+1}\sqrt{a + bX_t^2} \\ =& Z_{t+1}\sqrt{a + bZ_t^2(a + bX_{t-1}^2)} \\ =& Z_{t+1}\sqrt{a + bZ_t^2(a + bZ_{t-1}^2(a + bX_{t-2}^2))}\end{aligned}$$

   and so on to conclude that

$$X_{t+1} = Z_{t+1}\sqrt{a\left(1 + \sum_{k=1}^{\infty} b^k \prod_{j=1}^{k} Z_{t+1-j}^2\right)}$$

   It can be proved that the inside sum converges since the Z_t are all $N(0,1)$. (The expected value of the sum converges.) This gives a stationary process satisfying the model.