STAT 804: 2004-1

Assignment 5

1. Suppose X and Y are stationary independent processes with respective spectra f_X and f_Y. Compute the spectrum of $Z = aX + Y$.

   Solution We have

   $$
   \begin{aligned}
   &Cov(Z_t, Z_{t+h}) \\
   &\qquad = a^2 Cov(X_t, X_{t+h} + aCov(X_t, Y_{t+h}) + aCov(Y_t, X_{t+h}) + Cov(Y_t, Y_{t+h}) \\
   &\qquad = a^2 C_X(h) + C_Y(h)
   \end{aligned}
   $$

   where we use the independence to argue that two of the terms are 0. Multiplying by $\exp(2\pi\omega h i)$ and summing we get

   $$f_Z(\omega) = a^2 f_X(\omega) + f_Y(\omega)\,.$$

2. Suppose X and Y are jointly stationary processes and we observe them at times $1, \ldots, T$. Define the sample cross covariance $\hat{C}_{XY}(k) = \sum (X_t - \bar{X})(Y_{t+k} - \bar{Y})/T$ where are terms with index larger than T are interpreted as 0. Show that the sample cross covariance can be computed from the discrete Fourier transforms via

   $$\hat{C}_{XY}(m) = \sum_{k=0}^{T-1} \hat{X}(k)\overline{\hat{Y}(k)} \exp(2\pi i k m/T)/T$$

   (or figure out the correct formula).

   Solution: We have

   $$\sum_{k=0}^{T-1} \hat{X}(k)\overline{\hat{Y}(k)} \exp(2\pi i k m/T) = \sum_{k=0,s=0,t=0}^{T-1} \exp(2\pi i k(m+s-t)/T) X_s Y_t$$

   The sum over k gives either T or 0 according to whether or not $m+s-t$ is or is not an integer multiple of T. For m between $-(T-1)$ and $T-1$ the only possible non zero values arise when $m+s-t = -T$ or $m+s-t = 0$ or $m+s-t = T$. Fix a value of m. There are $T-m$ pairs s, t with $t - s = m$. For $m > 0$ there are m further pairs where

$m + s - t = T$ while for $m < 0$ there are m pairs with $m + s - t = -T$. This shows that

$$\sum_{k=0}^{T-1} \hat{X}(k)\overline{\hat{Y}(k)} \exp(2\pi ikm/T) = T \sum_{s=0}^{T-1} X_s Y_{t+m}$$

where indices larger than $T - 1$ have T subtracted from them. Thus the inverse transform produces a rather modified notion of the sample cross covariance.

3. Derive the frequency response function for the recursive filter

$$Y_t = aY_{t-1} + X_t$$

and plot the modulus squared and argument of the result for $a = 0.8$ and $a = 0.1$.

Solution: This filter can be written explicitly as

$$Y_t = \sum_0^\infty a^k X_{t-k}$$

so that the frequency response function is

$$A(\omega) = \sum_0^\infty a^k \exp(-2\pi\omega ki) = \frac{1}{1 - a\exp(-2\pi\omega i)}$$

This can be rationalized by multiplying by $1 - a\exp(2\pi\omega i)$ to get

$$A(\omega) = \frac{(1 - a\cos(2\pi\omega)) - ia\sin(2\pi\omega)}{1 + a^2 - 2a\cos(2\pi\omega)}$$

The power response function is then

$$|A(\omega)|^2 = \frac{(1 - a\cos(2\pi\omega))^2 + a^2\sin^2(2\pi\omega)}{(1 + a^2 - 2a\cos(2\pi\omega))^2} = \frac{1}{1 + a^2 - 2a\cos(2\pi\omega)}$$

while the phase $\phi(\omega)$ is defined by

$$\tan^{-1}\left(\frac{\sin(2\pi\omega)}{1 - a\cos(2\pi\omega)}\right).$$

(Note that here I am using the fact that the real part of A is positive because $|a| < 1$; this guarantees that the phase is between $-\pi/2$ and $\pi/2$ so we may use the arc tangent to compute ϕ.)

4. Compute and plot estimates of the spectrum for the time series `fake` for varying degrees of smoothing and compare the result to the spectrum of your fitted ARIMA model.

5. Let ϵ_t be a Gaussian white noise process. Define

$$X_t = \epsilon_{t-2} + 4\epsilon_{t-1} + 6\epsilon_t + 4\epsilon_{t+1} + \epsilon_{t+2}.$$

   Compute and plot the spectrum of X.

   Solution: This defines X as a filter applied to noise so that

$$f_X(\omega) = \sigma_\epsilon^2 |e^{-4\pi\omega i} + 4e^{-2\pi\omega i} + 6 + 4e^{2\pi\omega i} + e^{4\pi\omega i}|^2$$

   which becomes just

$$f_X(\omega) = \sigma_\epsilon^2 (2\cos(4\pi\omega) + 8\cos(2\pi\omega) + 6)^2$$

   which you could plot for say $\sigma = 1$.

6. For the filters A: $y_t = x_t - x_{t-12}$, B: $y_t = x_t - x_{t-1}$ and C defined by applying A then B determine the power transfer functions, plot them and interpret their effect on a spectrum. What is the effect of these filters on seasonal series? (Consider what the spectrum of a series with a strong seasonal effect is like.)

   Solution: We have

$$A_A(\omega) = 1 + e^{24\pi\omega i}$$
$$A_B(\omega) = 1 + e^{2\pi\omega i}$$

   and

$$A_C(\omega) = (1 + e^{24\pi\omega i})(1 + e^{2\pi\omega i}).$$

   You should probably make two plots: one of the gains of the filters and the other of the phase shifts. A seasonal series will have a spike in the periodogram at a frequency of 1/12 cycles per month or 1 cycle per year. The filter A has a frequency response which is 0 at $\omega = 1/12$ so the filter A removes that peak. Plugging in $\omega = 1/12$ in A_B gives a real part of 1.866 and an imaginary part of 0.5 for a gain of 1.93. The peak at 1 cycle per year is slightly magnified. Finally the filter C removes the peak at 1 cycle per year.