STAT 830
Independence and Conditioning

Richard Lockhart

Simon Fraser University

STAT 830 — Fall 2011

Purposes of These Notes

- Define independent events and random variables.
- Give conditions for independence.
- Define conditional probability, conditional distribution.
- State Bayes Theorem in various forms.

Independent Events pp 8-10

Def'n: Events A and B are independent if

$$P(AB) = P(A)P(B).$$

(Notation: AB is the event that both A and B happen, also written $A \cap B$.)

Def'n: A_i, $i = 1, \ldots, p$ are **independent** if

$$P(A_{i_1} \cdots A_{i_r}) = \prod_{j=1}^{r} P(A_{i_j})$$

for any $1 \le i_1 < \cdots < i_r \le p$.

Example: $p = 3$

$$\begin{aligned} P(A_1A_2A_3) &= P(A_1)P(A_2)P(A_3) \\ P(A_1A_2) &= P(A_1)P(A_2) \\ P(A_1A_3) &= P(A_1)P(A_3) \\ P(A_2A_3) &= P(A_2)P(A_3) \end{aligned}$$

All these equations needed for independence!

Counterexample

- Pairwise independence is not independence.
- Toss a coin twice.

$$A_1 = \{\text{first toss is a Head}\}$$
$$A_2 = \{\text{second toss is a Head}\}$$
$$A_3 = \{\text{first toss and second toss different}\}$$

- Then $P(A_i) = 1/2$ for each i and for $i \neq j$

$$P(A_i \cap A_j) = \frac{1}{4}$$

but

$$P(A_1 \cap A_2 \cap A_3) = 0 \neq P(A_1)P(A_2)P(A_3)\,.$$

Def'n: X and Y are **independent** if

$$P(X \in A; Y \in B) = P(X \in A)P(Y \in B)$$

for all A and B.

Notation: Write $X \perp\!\!\!\perp Y$.

Def'n: Rvs $X_1, \ldots, X_p$ **independent**:

$$P(X_1 \in A_1, \cdots, X_p \in A_p) = \prod P(X_i \in A_i)$$

for any $A_1, \ldots, A_p$.

Factorization criteria: Theorems 2.30, 2.33

Theorem

1. *If X and Y are independent then for all x, y*

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)\,.$$

2. *If X and Y are independent with joint density $f_{X,Y}(x,y)$ then X and Y have densities f_X and f_Y, and*

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\,.$$

3. *If X and Y independent with marginal densities f_X and f_Y then (X,Y) has joint density*

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\,.$$

Theorem Continued

Theorem (Theorem Continued)

4. *If* $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ *for* **all** x, y *then* X *and* Y *are independent.*

5. *If* (X, Y) *has density* $f(x, y)$ *and there exist* $g(x)$ *and* $h(y)$ *st* $f(x, y) = g(x)h(y)$ *for (almost)* **all** (x, y) *then* X *and* Y *are independent with densities given by*

$$f_X(x) = g(x) / \int_{-\infty}^{\infty} g(u) du$$

$$f_Y(y) = h(y) / \int_{-\infty}^{\infty} h(u) du \, .$$

6. *An analogous assertion to the previous holds in the discrete case.*

Proof of First Assertion

- Since X and Y are independent the events $X \le x$ and $Y \le y$ are independent
- So

$$P(X \le x, Y \le y) = P(X \le x)P(Y \le y)\,.$$

Proof of second assertion

- Suppose X and Y real valued.
- Asst 2: existence of $f_{X,Y}$ implies that of f_X and f_Y (marginal density formula).
- Then for any sets A and B

$$P(X \in A, Y \in B) = \int_A \int_B f_{X,Y}(x,y)dydx$$

$$P(X \in A)P(Y \in B) = \int_A f_X(x)dx \int_B f_Y(y)dy$$

$$= \int_A \int_B f_X(x)f_Y(y)dydx\,.$$

- Since $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$

$$\int_A \int_B [f_{X,Y}(x,y) - f_X(x)f_Y(y)]dydx = 0\,.$$

Measure theory shows quantity in [] is 0 for almost every pair (x,y).

Proof of third assertion

- For any A and B we have

$$\begin{aligned} P(X \in A, Y \in B) &= P(X \in A)P(Y \in B) \\ &= \int_A f_X(x)dx \int_B f_Y(y)dy \\ &= \int_A \int_B f_X(x)f_Y(y)dydx\,. \end{aligned}$$

  If we **define** $g(x, y) = f_X(x)f_Y(y)$ then we have proved that for $C = A \times B$

$$P((X, Y) \in C) = \int_C g(x, y)dydx\,.$$

- To prove that g is $f_{X,Y}$ prove this integral formula is valid for arbitrary Borel set C, not just rectangle $A \times B$.
- Use *monotone class* argument. Study closure properties collection of sets C for which identity holds.

Proof of fourth and fifth assertions

- For fourth assertion another monotone class argument.
- For fifth assertion:

$$\begin{aligned} P(X \in A, Y \in B) &= \int_A \int_B g(x)h(y)dydx \\ &= \int_A g(x)dx \int_B h(y)dy\,. \end{aligned}$$

  Take $B = R^1$ to see that

$$P(X \in A) = c_1 \int_A g(x)dx$$

  where $c_1 = \int h(y)dy$.
- So $c_1 g$ is the density of X. Since $\int\int f_{X,Y}(xy)dxdy = 1$ we see that $\int g(x)dx \int h(y)dy = 1$ so that $c_1 = 1/\int g(x)dx$.
- Similar argument for Y.

Inheritance of transformations

Theorem

If $X_1, \ldots, X_p$ are independent and $Y_i = g_i(X_i)$ then $Y_1, \ldots, Y_p$ are independent. Moreover, $(X_1, \ldots, X_q)$ and $(X_{q+1}, \ldots, X_p)$ are independent. (In fact everything you would expect to hold does.)

Def'n: $P(A|B) = P(AB)/P(B)$ if $P(B) \neq 0$.

Def'n: For discrete X and Y the conditional probability mass function of Y given X is

$$\begin{aligned} f_{Y|X}(y|x) &= P(Y = y|X = x) \\ &= f_{X,Y}(x, y)/f_X(x) \\ &= f_{X,Y}(x, y)/\sum_t f_{X,Y}(x, t) \end{aligned}$$

Conditional probability, continuous case p 37,38

- For absolutely continuous X $P(X = x) = 0$ for all x.
- What is $P(A|X = x)$ or $f_{Y|X}(y|x)$?
- Solution: use limit

$$P(A|X = x) = \lim_{\delta x \to 0} P(A|x \le X \le x + \delta x)$$

- If, e.g., X, Y have joint density $f_{X,Y}$ then with $A = \{Y \le y\}$ we have

$$P(A|x \le X \le x + \delta x) = \frac{P(A \cap \{x \le X \le x + \delta x\})}{P(x \le X \le x + \delta x)}$$

$$= \frac{\int_{-\infty}^{y} \int_{x}^{x+\delta x} f_{X,Y}(u, v) du dv}{\int_{x}^{x+\delta x} f_X(u) du}$$

- Divide top, bottom by δx; let $\delta x \to 0$.
- Denom converges to $f_X(x)$; numerator converges to

$$\int_{-\infty}^{y} f_{X,Y}(x, v) dv$$

Continuous case continued

- Define conditional cdf of Y given $X = x$:

$$P(Y \le y | X = x) = \frac{\int_{-\infty}^{y} f_{X,Y}(x,v)dv}{f_X(x)}$$

- Differentiate wrt y to get def'n of conditional density of Y given $X = x$:

$$f_{Y|X}(y|x) = f_{X,Y}(x,y)/f_X(x)\,;$$

  in words "conditional = joint/marginal".

Bayes Theorem pp 12,176-177

- From $P(AB) = P(A|B)P(B) = P(B|A)P(A)$ get

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

- Statistical description of difference between $B \implies A$ and $A \implies B$.
- Density formulation

$$f_{X|Y} = \frac{f_{Y|X} f_X}{f_Y}$$

- Bayesians like to write

$$(x|y) = (y|x)(x)/(y)$$

with the parentheses indicating densities and the letters indicating variables.

Generalizations

- More general formulas arise like

$$P(ABCD) = P(A|BCD)P(B|CD)P(C|D)P(D)$$

- Also: if $A_1, \ldots, A_k$ *mutually exclusive and exhaustive* then

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{\sum_i P(B|A_j)P(A_j)}$$

- *Mutually exclusive* means pairwise disjoint and *exhaustive* means

$$\cup_1^k A_i = \Omega.$$

- The density formula is really analogous since integrals are limits of sums

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_{Y|X}(y|x)f_X(x)}{\int_u f_{XY}(u,y)du}.$$