Lecture 10: MVN & Likelihood Asymptotics

- Properties of MVN:
- *Linear transforms of normals are normal*: If $X \sim MVN(\mu, \Sigma)$ and $Y = CX + \nu$ then $Y \sim MVN(C\mu + \nu, C\Sigma C^T)$.
- *All margins are normal*: if

$$X = \left[\begin{array}{c} X_1 \\ X_2 \end{array} \right] \sim MVN\left(\left[\begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right], X = \left[\begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array} \right] \right)$$

then

$$X_1 \sim MVN(\mu_1, \Sigma_{11}).$$

- And conditionals are normal

$$X_1 | X_2 = x_2 \sim MVN(\mu_1(x_2), \Sigma_{11 \cdot 2})$$

where

$$\mu_1(x) \equiv \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$

and

$$\Sigma_{11 \cdot 2} = \Sigma_{11} - \Sigma)12\Sigma_{22}^{-1}\Sigma_{21}.$$

MVN density

- Change of variables. Use the mnemonic

$$f_X(x)dx = f_Z(z)dz.$$

- Interpretation: $X, Z \in \mathbb{R}^p$ and Z has density $f_Z(z)$.
- Assume $X = g(Z)$ for a 1 to 1 differentiable function g. Then

$$f_X(x) = f_Z(z)\frac{dz}{dx}$$

means

$$f_X(x) = f_Z(g^{-1}(x))\left|\det\left(\frac{\partial g^{-1}(x)}{\partial x}\right)\right|$$

- The determinant term is called a *Jacobian*.

Polar co-ordinates example

- For instance in polar co-ordinates we might transform from rectangular (x, y) to (r, θ) so that $(r, \theta) = g(x, y)$ for some function g. We solve for x, y in terms of r, θ to get

$$x = r\cos\theta \quad y = r\sin\theta$$

and the Jacobian becomes

$$\left|\det\frac{\partial(x, y)}{\partial(r, \theta)}\right|$$

which is the absolute value of the determinant of

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$$

- We get $dxdy = rdrd\theta$.

Multivariate Normal density

- If $Z \sim MVN(0, I)$ and $X = AZ + \mu$ then $X \sim MVN(\mu, \mathring{A}^T)$.
- If A is invertible so is Σ and vice-versa.
- So g^{-1} is given by $A^{-1}(X - \mu)$.
- The joint density of Z is

$$(2\pi)^{-p} \exp\{-z^t z/2\}$$

- So the joint density of X is

$$f_Z(A^{-1}(x-\mu))|det(\partial z/\partial x)|$$

- The Jacobian is $|\det A|$.
- Density simplifies to

$$f_X(x) = (2\pi)^{-p/2} \exp(-(x-\mu)^T \Sigma^{-1}(x-\mu)/2)/\sqrt{\det(\Sigma)}.$$

Likelihood asymptotics

- I reiterated that

$$\frac{U_n(\theta_0)}{\sqrt{n}} \Longrightarrow MVN(0, \mathcal{I}_1(\theta_0)).$$

- Theorem: under regularity conditions:

$$\sqrt{n}(\hat{\theta}_n - \theta - 0) \Longrightarrow MVN(0, \mathcal{I}(\theta_0)^{-1}).$$

- Started proof based on

$$U_n(\hat{\theta}_n) = 0$$

  and Taylor expansion. You should look up multivariate Taylor formula.

Course coverage

- Chapter 2.11.
- Chapter 9.3-9.7, 9.10.