Lecture 12: Estimating Equations

- Motivating example: thermoluminescence dating of sediment.
- Samples of sand dune exposed to radiation
- Then heated in oven; they give off blue light – *thermoluminescence*.
- Set radiation doses $d_1, \ldots, d_n$ and measure corresponding amount of light $Y_1, \ldots, Y_n$.
- Reasonable model:

$$\mathrm{E}(Y_i) = \alpha + \beta d_i \equiv \mu_i$$

and

$$\mathrm{Var}(Y_i) \equiv \sigma_i^2 = \lambda \mu_i^2.$$

Lecture 12: Estimating Equations

- Might add the distributional assumption:

$$Y_i \sim N(\mu_i, \sigma_i^2).$$

- Several potential estimation methods: maximum likelihood, weighted least squares, iteratively re-weighted least squares.
- Maximum likelihood. Maximize

$$\ell(\alpha, \beta, \lambda) = -\frac{1}{2}\sum \frac{(Y_i - \mu_i)^2}{\sigma_i^2} - \sum \log \sigma_i.$$

- Solve likelihood equations which are

$$0 = \sum \frac{Y_i - \mu_i}{\sigma_i^2} - \sum \frac{\lambda}{\sigma_i}$$

$$0 = \sum \frac{(Y_i - \mu_i) d_i}{\sigma_i^2} - \sum \frac{\lambda d_i}{\sigma_i}$$

$$0 = \sum \frac{(Y_i - \mu_i)^2}{\lambda \sigma_i^2} - \sum \frac{1}{\lambda}.$$

Lecture 12: Estimating Equations

- Weighted least squares. Mimic regression and minimize

$$S(\alpha, \beta) = \frac{(Y_i - \mu_i)^2}{(\alpha + \beta d_i)^2}$$

  and then estimate λ separately.

- Take derivatives and solve

$$0 = \sum \frac{Y_i - \mu_i}{\sigma_i^2} - \sum \frac{\lambda}{\sigma_i} + \sum \frac{(Y_i - \mu_i)^2}{\sigma_i^3}$$

$$0 = \sum \frac{(Y_i - \mu_i) d_i}{\sigma_i^2} - \sum \frac{\lambda d_i}{\sigma_i} + \sum \frac{(Y_i - \mu_i)^2 d_i}{\sigma_i^3}$$

- Final method, IRWLS, is an iterative scheme.
- Start with preliminary estimates $\alpha^{(0)}, \beta^{(0)}$.

Lecture 12: Estimating Equations

- Find new estimates $\alpha^{(0)}, \beta^{(0)}$ by minimizing

$$\sum \frac{(Y_i - \alpha - \beta d_i)^2}{(\alpha^{(0)} + \beta^{(0)} d_i)^2}$$

  over α and β.

- The derivatives wrt α and β are

$$-\frac{1}{2} \sum \frac{Y_i - \alpha - \beta d_i}{(\alpha^{(0)} + \beta^{(0)} d_i)^2}$$

  and

$$-\frac{1}{2} \sum \frac{(Y_i - \alpha - \beta d_i) d_i}{(\alpha^{(0)} + \beta^{(0)} d_i)^2}$$

- Setting these equal to 0 gives two linear equations in 2 unknowns which are easy to solve.
- Replace $\alpha^{(0)}, \beta^{(0)}$ by $\alpha^{(1)}, \beta^{(1)}$ and iterate to get a sequence $\alpha^{(0)}, \beta^{(0)}$ for $k = 0, 1, 2, \cdots$.

Lecture 12: Estimating Equations

- Suppose the sequence converges to a limit (it does, in practice).
- Call the limit $\tilde{\alpha}, \tilde{\beta}$.
- These estimates solve the equations

$$\sum \frac{Y_i - \alpha - \beta d_i}{(\alpha + \beta d_i)^2} = 0$$
$$\sum \frac{(Y_i - \alpha - \beta d_i) d_i}{(\alpha + \beta d_i)^2} = 0.$$

- This is a set of *estimating equations*. Look back for 2 more examples in the same problem.
- The estimating equation $G(X, \theta) = 0$ is *unbiased* if

$$\mathrm{E}_\theta(G(X, \theta)) \equiv 0.$$

- MLE and IRWLS equations are unbiased, WLS is not.
- So WLS does not give *consistent* estimates.

Coverage in the text

- Estimating equations are not really covered in the text.
- See “course notes” on web pages 116-118.