Lecture 17: Profile Likelihood Ratio

- Regular model, param $\theta = (\psi, \eta)$, log-likelihood $\ell(\psi, \eta)$.
- Profile likelihood for ψ is

$$\ell(\psi, \hat{\eta}(\psi))$$

- Tests and CIs for ψ based on

$$2\left[\ell(\hat{\psi}, \hat{\eta}) - \ell(\psi, \hat{\eta}(\psi))\right] \sim \chi^2_\nu$$

where

$$\nu = \dim(\psi) = \dim(\theta) - \dim(\eta).$$

- Proof uses Taylor expansion of ℓ in two models: the full model and the model in which $\psi = \psi_0$ is known.
- Get two possible estimates $(\hat{\psi}, \hat{\eta})$ and $(\psi_0, \hat{\eta}(\psi_0))$.
- Use

$$\hat{\theta} - \theta_0 = \begin{bmatrix} \hat{\psi} - \psi_0 \\ \hat{\eta} - \eta_0 \end{bmatrix} \approx \mathcal{I}^{-1} U$$

and

$$\hat{\eta}(\psi_0) - \eta_0 = \approx \mathcal{I}_{\eta\eta}{}^{-1} U_\eta$$

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- Here U and $\mathcal{I}$ have been *partitioned*.
- Partitioning means

$$\mathcal{I} = \begin{bmatrix} \mathcal{I}_{\psi\psi} & \mathcal{I}_{\psi\eta} \\ \mathcal{I}_{\eta\psi} & \mathcal{I}_{\eta\eta} \end{bmatrix}$$

  and

$$U = \begin{bmatrix} U_\psi \\ U_\eta \end{bmatrix}$$

- The subscripts indicate derivatives with respect to the components of the indicated subvector of θ.
- Partitioned matrices may be manipulated like ordinary matrices as long as their shapes *conform* – as long as the multiplications make sense.

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- Also use the expansions

$$\ell(\hat{\psi}, \hat{\eta}) - \ell(\psi_0, \eta_0) \approx \mathcal{I}(\hat{\theta} - \theta_0)/2$$

and

$$\ell(\psi_0, \hat{\eta}(\psi_0)) - \ell(\psi_0, \eta_0) \approx \mathcal{I}_{\eta\eta}(\hat{\eta}(\psi_0) - \psi_0)/2$$

- Assemble the two expansions and subtract to get

$$2\left\{\ell(\hat{\psi}, \hat{\eta}) - \ell(\psi_0, \hat{\eta}(\psi_0))\right\} \approx U^T M U$$

where

$$M = \mathcal{I}^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{I}_{\eta\eta}{}^{-1} \end{bmatrix}$$

- Then use previous theory about quadratic forms in normal variates.

Coverage in the text

- Chapter 10.
- Course slides "Hypothesis Tests": 1-9, 13, 14, 18-24, 26, 27
- See "course notes" on web pages 130-131.