Lecture 5: Approximate confidence intervals

- Types of convergence
- $X_n \to X$ in probability if for every $\epsilon > 0$:

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0.$$

- $X_n \to X$ almost surely (a.s.) if

$$P(\lim_{n\to\infty} X_n = X) = 1.$$

- $X_n \to X$ in pth mean ($p > 0$) if

$$\lim_{n\to\infty} \mathrm{E}(|X_n - X|^p) = 0.$$

- Almost sure or pth mean convergence implies convergence in probability implies convergence in distribution.
- If X is constant then convergence in distribution implies convergence in probability.

Application

- Use Slutsky's theorem with

$$X_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \Longrightarrow X \sim N(0,1) \quad Y_n = \frac{s}{\sigma}.$$

- Take $f(x,y) = x/y$.
- Use law of large numbers to conclude $Y_n \Longrightarrow 1$.
- Strong law of large numbers: if $X_1, X_2, \cdots$ are iid with $\mathrm{E}(|X_1|) < \infty$ then

$$\bar{X} \to \mu = \mathrm{E}(X_1) \quad \text{a.s.}$$

- Weak law – same conditions, conclusion is convergence in probability.
- Manipulate SLLN as usual: $\bar{X^2} \to \mathrm{E}(X_1^2) = \mu^2 + \sigma^2$ and $\bar{X} \to \mu$ so $s^2 \to \mu^2 + \sigma^2 - \mu^2 = \sigma^2$ so $s \to \sigma$ all almost surely.
- Conclude that

$$\frac{X_n}{Y_n} = t = \frac{\sqrt{n}(\bar{X} - \mu)}{s} \Longrightarrow N(0,1)$$

Standard Errors: exact, approximate, estimated

- If $\hat\phi$ is an estimate of ϕ the standard error of $\hat\phi$ is

$$\sigma_{\hat\phi} = \sqrt{\mathrm{Var}(\hat\phi)}.$$

- A sequence of parameter values $\sigma_{n,\hat\phi}$ gives an approximate standard error for $\hat\phi$ if

$$\frac{\hat\phi - \phi}{\sigma_{n,\hat\phi}} \Longrightarrow N(0,1).$$

- An estimated standard error $\hat\sigma)\hat\sigma_{n,\hat\phi}$ is a sequence of estimates of the approximate standard error.
- Estimated SE's are useful if

$$\frac{\sigma_{n,\hat\phi}}{\hat\sigma_{n,\hat\phi}} \to 1 \text{ in probability}$$

so that

$$\frac{\hat\phi - \phi}{\hat\sigma_{n,\hat\phi}} \Longrightarrow N(0,1).$$

Lecture 5 summary

- Then we get approximate level $1-\alpha$ confidence intervals from

$$\hat{\phi} \pm z_{\alpha/2}\hat{\sigma}_{n,\hat{\phi}}.$$

- Can get two different approximate confidence intervals for $F(x)$ from the two approximate pivots

$$\frac{\sqrt{n}(\hat{F}_n(x) - F(x))}{\sqrt{F(x)(1-F(x))}} \Longrightarrow N(0,1)$$

  and

$$\frac{\sqrt{n}(\hat{F}_n(x) - F(x))}{\sqrt{\hat{F}(x)(1-\hat{F}(x))}} \Longrightarrow N(0,1)$$

- In homework you solve the inequalities $-z_{\alpha/2} \le \text{pivot} \le z_{\alpha/2}$ to get $L \le F(x) \le U$ where L, U are *statistics*.
- You also work out simultaneous confidence intervals and study them by Monte Carlo.

Simultaneous confidence bands

- But *simultaneous* intervals often wanted or needed:

$$P(\forall x : L(X, x) \le F(x) \le U(X, x)) = 1 - \alpha$$

  gives a *simultaneous confidence band* $L(X, x)$ to $U(X, x)$ for the whole function $F(x)$.

- Conservative but with guaranteed coverage from DKW inequality.

$$P(\exists x : \sqrt{n}|\hat{F}_n(x) - F(x)| > \sqrt{-\log(\alpha/2)/2}) \le \alpha.$$

- Based on *weak convergence* (convergence in distribution for random functions):

$$P(\exists x : \sqrt{n}|\hat{F}_n(x) - F(x)| > u) \to P(\exists t : |B_0(t)| > u)$$

  where B_0 is a *Brownian Bridge*.

Course coverage

- Chapter 7.1
- Chapter 5.1-4.