STAT 801=830
Generating Functions

Richard Lockhart

Simon Fraser University

STAT 801=830 — Fall 2012

What I think you already have seen

- Definition of Moment Generating Function
- Basics of complex numbers

What I want you to learn

- Definition of cumulants and cumulant generating function.
- Definition of Characteristic Function
- Elementary features of complex numbers
- How they “characterize” a distribution
- Relation to sums of independent rvs

Moment Generating Functions pp 56-58

- **Def'n**: The moment generating function of a real valued X is

$$M_X(t) = E(e^{tX})$$

defined for those real t for which the expected value is finite.

- **Def'n**: The moment generating function of $X \in R^p$ is

$$M_X(u) = E[e^{u^t X}]$$

defined for those vectors u for which the expected value is finite.

- Formal connection to moments:

$$\begin{aligned} M_X(t) &= \sum_{k=0}^{\infty} E[(tX)^k]/k! \\ &= \sum_{k=0}^{\infty} \mu_k' t^k/k! \,. \end{aligned}$$

- Sometimes can find power series expansion of M_X and read off the moments of X from the coefficients of $t^k/k!$.

Moments and MGFs

[redacted]

If M is finite for all $t \in [-\epsilon, \epsilon]$ for some $\epsilon > 0$ then

1. *Every moment of X is finite.*
2. *M is C^∞ (in fact M is analytic).*
3. $\mu'_k = \frac{d^k}{dt^k} M_X(0)$.

- Note: C^∞ means has continuous derivatives of all orders.
- Analytic means has convergent power series expansion in neighbourhood of each $t \in (-\epsilon, \epsilon)$.
- The proof, and many other facts about mgfs, rely on techniques of complex variables.

MGFs and Sums

- If $X_1, \ldots, X_p$ are independent and $Y = \sum X_i$ then the moment generating function of Y is the product of those of the individual X_i:

$$M_Y(t) = E(e^{tY}) = \prod_i E(e^{tX_i}) = \prod_i M_{X_i}(t).$$

- Note: also true for multivariate X_i.
- Problem: power series expansion of M_Y not nice function of expansions of individual M_{X_i}.
- Related fact: first 3 moments (meaning μ, σ^2 and μ_3) of Y are sums of those of the X_i:

$$\begin{aligned} E(Y) &= \sum E(X_i) \\ \mathrm{Var}(Y) &= \sum \mathrm{Var}(X_i) \\ E[(Y - E(Y))^3] &= \sum E[(X_i - E(X_i))^3] \end{aligned}$$

Cumulants *not in text*

- However:

$$E[(Y-E(Y))^4] = \sum \{E[(X_i - E(X_i))^4] - 3E^2[(X_i - E(X_i))^2]\} + 3\left\{\sum E[(X_i - E(X_i))^2]\right\}^2$$

- But related quantities: **cumulants** add up properly.
- Note: log of the mgf of Y is sum of logs of mgfs of the X_i.
- **Def'n**: the cumulant generating function of a variable X by

$$K_X(t) = \log(M_X(t)).$$

- Then

$$K_Y(t) = \sum K_{X_i}(t).$$

- Note: mgfs are all positive so that the cumulant generating functions are defined wherever the mgfs are.

Relation between cumulants and moments

- So: K_Y has power series expansion:

$$K_Y(t) = \sum_{r=1}^{\infty} \kappa_r t^r / r! \,.$$

- **Def'n**: the κ_r are the cumulants of Y.
- Observe

$$\kappa_r(Y) = \sum \kappa_r(X_i) \,.$$

- Cumulant generating function is

$$\begin{aligned} K(t) &= \log(M(t)) \\ &= \log(1 + [\mu_1 t + \mu_2' t^2/2 + \mu_3' t^3/3! + \cdots]) \end{aligned}$$

- Call quantity in $[\ldots]$ x; expand

$$\log(1+x) = x - x^2/2 + x^3/3 - x^4/4 \cdots \,.$$

Cumulants and moments

- Stick in the power series

$$x = \mu t + \mu_2' t^2/2 + \mu_3' t^3/3! + \cdots ;$$

- Expand out powers of x; collect together like terms.
- For instance,

$$\begin{aligned} x^2 &= \mu^2 t^2 + \mu\mu_2' t^3 + [2\mu_3'\mu/3! + (\mu_2')^2/4]t^4 + \cdots \\ x^3 &= \mu^3 t^3 + 3\mu_2'\mu^2 t^4/2 + \cdots \\ x^4 &= \mu^4 t^4 + \cdots . \end{aligned}$$

- Now gather up the terms.
- The power t^1 occurs only in x with coefficient μ.
- The power t^2 occurs in x and in x^2 and so on.

Cumulants and moments

- Putting these together gives

$$K(t) = \mu t + [\mu_2' - \mu^2]t^2/2 + [\mu_3' - 3\mu\mu_2' + 2\mu^3]t^3/3! + [\mu_4' - 4\mu_3'\mu - 3(\mu_2')^2 + 12\mu_2'\mu^2 - 6\mu^4]t^4/4! \cdots$$

- Comparing coefficients to $t^r/r!$ we see that

$$\begin{aligned}
\kappa_1 &= \mu \\
\kappa_2 &= \mu_2' - \mu^2 = \sigma^2 \\
\kappa_3 &= \mu_3' - 3\mu\mu_2' + 2\mu^3 = E[(X-\mu)^3] \\
\kappa_4 &= \mu_4' - 4\mu_3'\mu - 3(\mu_2')^2 + 12\mu_2'\mu^2 - 6\mu^4 \\
&= E[(X-\mu)^4] - 3\sigma^4 .
\end{aligned}$$

- Reference: Kendall and Stuart (or new version called *Kendall's Theory of Advanced Statistics* by Stuart and Ord) for formulas for larger orders r.

Example, N(0,1)

- **Example**: $X_1, \ldots, X_p$ independent, $X_i \sim N(\mu_i, \sigma_i^2)$:

$$\begin{aligned} M_{X_i}(t) &= \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}(x-\mu_i)^2/\sigma_i^2} dx/(\sqrt{2\pi}\sigma_i) \\ &= \int_{-\infty}^{\infty} e^{t(\sigma_i z+\mu_i)} e^{-z^2/2} dz/\sqrt{2\pi} \\ &= e^{t\mu_i} \int_{-\infty}^{\infty} e^{-(z-t\sigma_i)^2/2+t^2\sigma_i^2/2} dz/\sqrt{2\pi} \\ &= e^{\sigma_i^2 t^2/2+t\mu_i} . \end{aligned}$$

- So cumulant generating function is:

$$K_{X_i}(t) = \log(M_{X_i}(t)) = \sigma_i^2 t^2/2 + \mu_i t .$$

- Cumulants are $\kappa_1 = \mu_i$, $\kappa_2 = \sigma_i^2$ and every other cumulant is 0.
- Cumulant generating function for $Y = \sum X_i$ is

$$K_Y(t) = \sum \sigma_i^2 t^2/2 + t \sum \mu_i$$

which is the cumulant generating function of $N(\sum \mu_i, \sum \sigma_i^2)$.

Chi-squared distributions

- **Example**: Homework: derive moment and cumulant generating function and moments of a Gamma rv.
- Now suppose $Z_1, \ldots, Z_\nu$ independent $N(0,1)$ rvs.
- By definition: $S_\nu = \sum_1^\nu Z_i^2$ has χ^2_ν distribution.
- It is easy to check $S_1 = Z_1^2$ has density

$$(u/2)^{-1/2} e^{-u/2} / (2\sqrt{\pi})$$

  and then the mgf of S_1 is

$$(1-2t)^{-1/2}\,.$$

- It follows that

$$M_{S_\nu}(t) = (1-2t)^{-\nu/2}$$

  which is (homework) moment generating function of a Gamma$(\nu/2, 2)$ rv.
- SO: χ^2_ν dstbn has Gamma$(\nu/2, 2)$ density:

$$(u/2)^{(\nu-2)/2} e^{-u/2} / (2\Gamma(\nu/2))\,.$$

Cauchy Distribution

- **Example**: The Cauchy density is

$$\frac{1}{\pi(1+x^2)};$$

corresponding moment generating function is

$$M(t) = \int_{-\infty}^{\infty} \frac{e^{tx}}{\pi(1+x^2)} dx$$

which is $+\infty$ except for $t = 0$ where we get 1.

- *Every* t distribution has exactly same mgf.
- So: can't use mgf to distinguish such distributions.
- Problem: these distributions do not have infinitely many finite moments.
- So: develop substitute for mgf which is defined for every distribution, namely, the characteristic function.

Aside on complex arithmetic

- Complex numbers: add $i = \sqrt{-1}$ to the real numbers.
- Require all the usual rules of algebra to work.
- So: if i and any real numbers a and b are to be complex numbers then so must be $a + bi$.
- Multiplication: If we multiply a complex number $a + bi$ with a and b real by another such number, say $c + di$ then the usual rules of arithmetic (associative, commutative and distributive laws) require

$$\begin{aligned}(a+bi)(c+di) &= ac + adi + bci + bdi^2 \\ &= ac + bd(-1) + (ad+bc)i \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

so this is precisely how we define multiplication.

Complex aside, slide 2

- Addition: follow usual rules to get
$$(a+bi)+(c+di)=(a+c)+(b+d)i\,.$$
- Additive inverses: $-(a+bi)=-a+(-b)i$.
- Multiplicative inverses:
$$\begin{aligned}\frac{1}{a+bi} &= \frac{1}{a+bi}\frac{a-bi}{a-bi}\\ &= \frac{a-bi}{a^2-abi+abi-b^2i^2}=\frac{a-bi}{a^2+b^2}\,.\end{aligned}$$
- Division:
$$\frac{a+bi}{c+di}=\frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)}=\frac{ac-bd+(bc+ad)i}{c^2+d^2}\,.$$
- Notice: usual rules of arithmetic don't require any more numbers than
$$x+yi$$
where x and y are real.

Complex Aside Slide 3

- **Transcendental functions**: For real x have $e^x = \sum x^k/k!$ so

$$e^{x+iy} = e^x e^{iy} .$$

- How to compute e^{iy}?
- Remember $i^2 = -1$ so $i^3 = -i$, $i^4 = 1$ $i^5 = i^1 = i$ and so on. Then

$$\begin{aligned}
e^{iy} &= \sum_0^\infty \frac{(iy)^k}{k!} \\
&= 1 + iy + (iy)^2/2 + (iy)^3/6 + \cdots \\
&= 1 - y^2/2 + y^4/4! - y^6/6! + \cdots \\
&\quad + iy - iy^3/3! + iy^5/5! + \cdots \\
&= \cos(y) + i\sin(y)
\end{aligned}$$

- We can thus write

$$e^{x+iy} = e^x(\cos(y) + i\sin(y))$$

Complex Aside Slide 4, Argand diagrams

- Identify $x + yi$ with the corresponding point (x, y) in the plane.
- Picture the complex numbers as forming a plane.
- Now every point in the plane can be written in polar co-ordinates as $(r\cos\theta, r\sin\theta)$ and comparing this with our formula for the exponential we see we can write

$$x + iy = \sqrt{x^2 + y^2}\, e^{i\theta} = re^{i\theta}$$

  for an angle $\theta \in [0, 2\pi)$.
- Multiplication revisited: $x + iy = re^{i\theta}$, $x' + iy' = r'e^{i\theta'}$.

$$(x + iy)(x' + iy') = re^{i\theta} r' e^{i\theta'} = rr' e^{i(\theta+\theta')} .$$

Complex Aside Slide 4, Argand diagrams

- We will need from time to time a couple of other definitions:
- **Definition**: The **modulus** of $x + iy$ is

$$|x + iy| = \sqrt{x^2 + y^2}.$$

- **Definition**: The **complex conjugate** of $x + iy$ is $\overline{x + iy} = x - iy$.
- Some identities: $z = x + iy = re^{i\theta}$ and $z' = x' + iy' = r'e^{i\theta'}$.
- Then

$$z\overline{z} = x^2 + y^2 = r^2 = |z|^2$$

$$\frac{z'}{z} = \frac{z'\overline{z}}{|z|^2} = rr'e^{i(\theta' - \theta)}$$

$$\overline{re^{i\theta}} = re^{-i\theta}.$$

Notes on calculus with complex variables

- Essentially usual rules apply so, for example,

$$\frac{d}{dt}e^{it} = ie^{it} .$$

- We will (mostly) be doing only integrals over the real line; the theory of integrals along paths in the complex plane is a very important part of mathematics, however.
- FACT: (not used explicitly in course). If $f : \mathbb{C} \mapsto \mathbb{C}$ is differentiable then f is analytic (has power series expansion).

End of Aside

Characteristic Functions

- **Def'n**: The characteristic function of a real rv X is

$$\phi_X(t) = E(e^{itX})$$

  where $i = \sqrt{-1}$ is the imaginary unit.

- Since

$$e^{itX} = \cos(tX) + i\sin(tX)$$

  we find that

$$\phi_X(t) = E(\cos(tX)) + iE(\sin(tX)) .$$

- Since the trigonometric functions are bounded by 1 the expected values must be finite for all t.
- This is precisely the reason for using characteristic rather than moment generating functions in probability theory courses.

Role of transforms in characterization cf Th 3.33, p 57

[redacted]

For any two real rvs X and Y the following are equivalent:

1. *X and Y have the same distribution, that is, for any (Borel) set A we have*
$$P(X \in A) = P(Y \in A)\,.$$
2. *$F_X(t) = F_Y(t)$ for all t.*
3. *$\phi_X(t) = E(e^{itX}) = E(e^{itY}) = \phi_Y(t)$ for all real t.*

Moreover, all these are implied if there is $\epsilon > 0$ such that for all $|t| \le \epsilon$

$$M_X(t) = M_Y(t) < \infty\,.$$