STAT 801=830
Convergence of RVs

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Purposes of These Notes

- Define convergence in probability, in mean, in quadratic mean.
- Define almost sure convergence.
- Define convergence in distribution.
- Compare and contrast these definitions.

Intuition about convergences pp 71-72

- Contrast two statements:
  - X and Y are close together.
  - X and Y have similar distributions.
- Truth of first statement depends on *joint* distribution of X and Y.
- Truth of second statement depends on *marginal* distributions of X and Y.
- Both ideas used in *large sample theory*: describing behaviour of statistical procedures *approximately* in presence of lots of data.

Relation between convergence and approximation

- Some approximations and the limits they come from.
- Stirling's approximation:

$$n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n} \equiv s_n$$

n	$n!$	s_n	$n!/s_n$
5	120	118.019	1.01678
10	3628800	3598695.619	1.008365

- Normal approximation to Binomial. Toss coin 100 times, get X heads.

$$P(40 \le X \le 60) \approx \Phi(\frac{60-50}{\sqrt{25}}) - \Phi(\frac{40-50}{\sqrt{25}}) = 0.9544997.$$

- Same context better approximation

$$P(40 \le X \le 60) \approx \Phi(\frac{60.5-50}{\sqrt{25}}) - \Phi(\frac{39.5-50}{\sqrt{25}}) = 0.9642712$$

- Same context

$$P(X = 50) \approx ?$$

Associated Limits

- Theorem:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}n^{n+1/2}e^{-n}} = 1.$$

- Used when $n = 100$ to give, for instance,

$$100! \approx \sqrt{2\pi 100}100^{100}e^{-100} = 9.3326215443944152682 \times 10^{157}$$

- All those digits are right!
- Theorem: if $X_n \sim \mathrm{Binomial}(n, 1/2)$ then

$$\lim_{n\to\infty} P\left(\frac{X_n - n/2}{\sqrt{n/4}} \le x\right) = \Phi(x)$$

- Used with $x\sqrt{100/4} + 100/2$ equal to 60 and 40 or 60.5 and 39.5.
- Used with 49.5 and 50.5 to get $P(X_{100} = 50)$ approximately.

Summary

- If we want to compute x_{100} we compute $y = \lim_{n\to\infty} x_n$ and approximate $x_{100} \approx y$.
- There are often many different ways to think of x_{100} as an entry in some sequence! Get slightly different approximations.
- And some of the approximations are lousy; some are great.

Limits of Random Variables NOT Distributions

- We do an experiment to measure probability that a dropped tack lands point up.
- Drop tack n times, observe $X_n \sim \mathrm{Binomial}(n, p)$, which is number of times tack lands point up.
- Two common random variables to study:

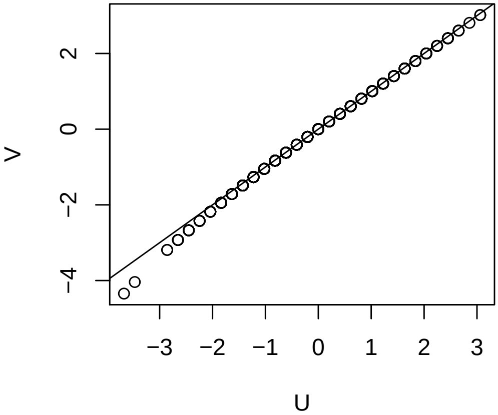
$$U_n \equiv \frac{X_n - np}{\sqrt{np(1-p)}} \quad \text{and} \quad V_n \equiv \frac{X_n - np}{\sqrt{n\hat{p}(1-\hat{p})}}$$

  where $\hat{p} = X/n$.
- First used in testing, second to form confidence intervals.
- Approximation is

$$U_n \approx V_n$$

- Generate 1000 values of X with $p = 0.4$. Plot U vs V.

Effect of estimating standard error



Convergence in Probability and Almost Surely pp 72-75,81

- In this case the plot shows

$$P(|U_n - V_n| \text{ is big})$$

  is small. (The points are close to the line $y = x$.)

- **Def'n**: A sequence of random variables X_n *converges in probability* to a random variable X if for every $\epsilon > 0$ we have

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0.$$

- **Def'n**: A sequence of random variables X_n *converges almost surely* to a random variable X if

$$P(\lim_{n\to\infty} X_n = X) = 1.$$

Convergence in pth mean especially quadratic

- **Def'n**: A sequence of random variables X_n *converges in mean* or *converges in L_1* to a random variable X if

$$\lim_{n\to\infty} \mathrm{E}(|X_n - X|) = 0.$$

- **Def'n**: A sequence of random variables X_n *converges in quadratic mean* or *converges in L_2* to a random variable X if

$$\lim_{n\to\infty} \mathrm{E}(|X_n - X|^2) = 0.$$

- For pth mean we use

$$\lim_{n\to\infty} \mathrm{E}(|X_n - X|^p) = 0.$$

Back to our example

- In fact $U_n - V_n$ converges to 0 in probability.
- And $U_n - V_n$ converges to 0 almost surely.
- But they do not converge in pth mean because V_n does not have a finite mean ($P(\hat{p} = 0) > 0$).