

# STAT 801=830

## The Multivariate Normal Distribution

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STAT 801=830 — Fall 2012



# What I assume you already know

- The basics of normal distributions in 1 dimension.



# The Multivariate Normal Distribution

- **Def'n:**  $Z \in R^1 \sim N(0, 1)$  iff

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

- **Def'n:**  $Z \in R^p \sim MVN(0, I)$  if and only if  $Z = (Z_1, \dots, Z_p)^t$  with the  $Z_i$  independent and each  $Z_i \sim N(0, 1)$ .
- In this case according to our theorem

$$\begin{aligned} f_Z(z_1, \dots, z_p) &= \prod \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} \\ &= (2\pi)^{-p/2} \exp\{-z^t z/2\}; \end{aligned}$$

superscript  $t$  denotes matrix transpose.

- **Def'n:**  $X \in R^p$  has a multivariate normal distribution if it has the same distribution as  $AZ + \mu$  for some  $\mu \in R^p$ , some  $p \times p$  matrix of constants  $A$  and  $Z \sim MVN(0, I)$ .



# The Multivariate Normal Density

- Matrix  $A$  singular:  $X$  does not have a density.
- $A$  invertible: derive multivariate normal density by change of variables:

$$X = AZ + \mu \Leftrightarrow Z = A^{-1}(X - \mu) \quad \frac{\partial X}{\partial Z} = A \quad \frac{\partial Z}{\partial X} = A^{-1}.$$

- So

$$\begin{aligned} f_X(x) &= f_Z(A^{-1}(x - \mu)) |\det(A^{-1})| \\ &= \frac{\exp\{-(x - \mu)^t (A^{-1})^t A^{-1} (x - \mu) / 2\}}{(2\pi)^{p/2} |\det A|}. \end{aligned}$$



## The Multivariate Normal Density continued

- Now define  $\Sigma = AA^t$  and notice that

$$\Sigma^{-1} = (A^t)^{-1}A^{-1} = (A^{-1})^tA^{-1}$$

and

$$\det \Sigma = \det A \det A^t = (\det A)^2.$$

- Thus  $f_X$  is

$$\frac{\exp\{-(x - \mu)^t \Sigma^{-1} (x - \mu)/2\}}{(2\pi)^{p/2} (\det \Sigma)^{1/2}};$$

the  $MVN(\mu, \Sigma)$  density.

- Note density is the same for all  $A$  such that  $AA^t = \Sigma$ .
- This justifies the notation  $MVN(\mu, \Sigma)$ .



## The Multivariate Normal Density continued

- For which  $\mu$ ,  $\Sigma$  is this a density?
- Any  $\mu$  but if  $x \in R^p$  then, putting  $y = A^t x$ ,

$$x^t \Sigma x = x^t A A^t x = (A^t x)^t (A^t x) = \sum_1^p y_i^2 \geq 0$$

- Inequality strict except for  $y = 0$  which is equivalent to  $x = 0$ .
- Thus  $\Sigma$  is a positive definite symmetric matrix.
- Conversely, if  $\Sigma$  is a positive definite symmetric matrix then there is a square invertible matrix  $A$  such that  $A A^t = \Sigma$  so that there is a  $MVN(\mu, \Sigma)$  distribution.
- $A$  can be found via the Cholesky decomposition, e.g.



## Singular cases

- When  $A$  is singular  $X$  will not have a density.
- $\exists a$  such that  $P(a^t X = a^t \mu) = 1$
- $X$  is confined to a hyperplane.
- Still true: distribution of  $X$  depends only on  $\Sigma = AA^t$
- if  $AA^t = BB^t$  then  $AZ + \mu$  and  $BZ + \mu$  have the same distribution.
- Proof by mgfs in case  $\mu = 0$ :

$$E(\exp(u^t AZ)) = E(\exp(v^t Z))$$

where  $v = A^t u$ . Use the independent components of  $Z$  to get

$$E(\exp(u^t AZ)) = \exp(v^t v / 2) = \exp(u^t AA^t u) = \exp(u^t BB^t u) = E(\exp(u^t AZ))$$



# Properties of the *MVN* distribution

- ① All margins are multivariate normal: if

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

then  $X \sim MVN(\mu, \Sigma) \Rightarrow X_1 \sim MVN(\mu_1, \Sigma_{11})$ .

- ② All conditionals are normal: the conditional distribution of  $X_1$  given  $X_2 = x_2$  is  $MVN(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$
- ③  $MX + \nu \sim MVN(M\mu + \nu, M\Sigma M^t)$ : affine transformation of MVN normal.

