Lecture 16: Cochran's Theorem

- If $Z \sim \text{MVN}_n(0, I)$ and Q is symmetric and idempotent then

$$Z^T QZ \sim \chi^2_\nu \text{ where } \nu = \text{tr}(Q).$$

- If Q_1 and Q_2 are each symmetric and idempotent and $Q_1 Q_2 = 0$ then $Z^T Q_1 Z$ and $Z^T Q_2 Z$ are independent.
- If $X = [X_1^T \quad X_2^T]^T$ is $\text{MVN}_n(0, \Sigma)$ and $\text{Cov}(X_1, X_2) = 0$ then X_1 and X_2 are independent.
- I proved that by proving that the joint characteristic function of X_1 and X_2 is the same as the joint characteristic function of two independent variables.
- I illustrated the points with regression:

$$Y = X\beta + \epsilon = X\beta + \sigma Z$$

where Z is standard multivariate normal.

Lecture 16: ANOVA tables

- The fitted vector $\hat{Y}$ is given by

  $$\hat{Y} = X\hat{\beta} = X(X^TX)^{-1}X^TY = X(X^TX)^{-1}X^T(X\beta+\sigma Z) = \beta+\sigma HZ$$

  where $H = X(X^TX)^1X^T$ is hat matrix. Note that $\hat{Y} = HY$.
- The matrix H is idempotent.
- The residual vector is $\hat{\epsilon} = \sigma\hat{Z} = Y - \hat{Y} = (I-H)Y$.
- These two vectors are orthogonal:

  $$\hat{Y}^T\hat{\epsilon} = Y^TH^T(I-H)Y = Y^TH(I-H)Y = Y(H-H)Y = 0.$$

- So by Pythagoras' theorem:

  $$Y^TY = \hat{Y}^T\hat{Y} + \hat{\epsilon}^T\hat{\epsilon}$$

- This is an ANOVA identity: Total SS = Regression SS + Error SS where SS stands for Sum of Squares.
- Usually presented in an ANOVA table with columns source, df, Mean Square, F, and P.
- By Cochran's theorem RSS and ESS are independent.

The Multivariate Normal Density

- Change of variables. Use the mnemonic

$$f_X(x)dx = f_Z(z)dz.$$

- Interpretation: $X, Z \in \mathbb{R}^p$ and Z has density $f_Z(z)$.
- Assume $X = g(Z)$ for a 1 to 1 differentiable function g. Then

$$f_X(x) = f_Z(z)\frac{dz}{dx}$$

means

$$f_X(x) = f_Z(g^{-1}(x))\left|\det\left(\frac{\partial g^{-1}(x)}{\partial x}\right)\right|$$

- The determinant term is called a *Jacobian*.
- So if X is MVN(μ, Σ) with Σ non singular then $Z = AZ + \mu$ where $AA^T = \Sigma$.
- The Jacobian is $|\det A^{-1}| = (\det \Sigma)^{-1/2}$.
- The density is

$$f_X(x) = (2\pi)^{n/2}(\det \Sigma)^{-1/2}\exp(-(x-\mu)^T\Sigma^{-1}(x-\mu)/2).$$

Coverage in the text

- TBD
- Course slides “Multivariate Normal”,
- Support material “Samples from the Normal Distribiution”.
- See “course notes” on web pages 52-58,