Lecture 9: Likelihood Asymptotics

- I stated Jensen's inequality
- I proved that if θ is identifiable then

$$\frac{\ell(\theta_0) - \ell(\theta)}{n} \to \mu(\theta) \geq 0$$

  with a strict inequality except at $\theta = \theta_0$.
- Above two apply to multivariate parameters.
- I started to prove: in the Cauchy problem the convergence is uniform on some closed interval around θ_0.
- So if $\hat{\theta}_n$ maximizes the likelihood over this interval then

$$\hat{\theta}_n \to \theta_0$$

  almost surely; we say $\hat{\theta}_n$ is strongly *consistent*.

A lemma

Lemma

Suppose

1. f *is a continuous function on some compact set* K
2. f *has a unique maximum at* $x_0 \in K$
3. f_n *is a sequence of functions converging uniformly to* f *on* K
4. x_n *maximizes* f_n

Then $\lim_{n\to\infty} x_n = x_0$.

Proof

Uniform convergence means

$$lim_{n\to\infty} \sup_{x\in K} |f_n(x) - f(x)| = 0.$$

So

$$lim_{n\to\infty} |f_n(x_n) - f(x_n)| = 0.$$

It follows that

$$f(x_0) \geq \limsup_{n\to\infty} f(x_n) \geq \liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} f_n(x_n) \geq \liminf_{n\to\infty} f_n(x_0) = f(x_0)$$

Hence

$$\lim_{n\to\infty} f(x_n) = f(x_0).$$

Since f is continuous and x_0 is the unique maximizer we have

$$\lim_{n\to\infty} x_n = x_0.$$

Further details of the last bit

If not there is a subsequence of x_n converging to some $x^* \neq x_0$ (by compactness or the Heine-Borel theorem). For this subsequence

$$\lim f(x_n) \to f(x^*) < f(x_0) = \lim f(x_n)$$

which is a contradiction.
Heine-Borel says every bounded sequence has a convergent subsequence.
To apply it you say that if the sequence x_n did not converge to x_0 there would be an $\epsilon > 0$ such that

$$|x_n - x_0| \geq \epsilon$$

for infinitely many n. This infinite set of n defines a subsequence of the original sequence to which Heine-Borel is applied to find x^* with $|x^* - x_0| \geq \epsilon$.

Course coverage

- Chapter 9.3-7.
- Chapter 2.10.