STAT 830
Confidence Sets

Richard Lockhart

Simon Fraser University

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Purposes of These Notes

- Discuss exact and approximate confidence intervals.
- Discuss role of pivotals in finding confidence intervals.

Confidence Intervals

- **Definition**: A level β confidence set for a parameter $\phi(\theta)$ is a random subset C, of the set of possible values of ϕ such that for each θ

$$P_\theta(\phi(\theta) \in C) \geq \beta$$

- Confidence sets are very closely connected with hypothesis tests:
- First from confidence sets to hypothesis tests.
- Suppose C is a level $\beta = 1 - \alpha$ confidence set for ϕ.
- To test $\phi = \phi_0$: reject if $\phi \notin C$.
- This test has level α.

From tests to confidence sets

- Conversely, suppose that for each ϕ_0 we have available a level α test of $\phi = \phi_0$ whose rejection region is say R_{ϕ_0}.
- Define $C = \{\phi_0 : \phi = \phi_0 \text{ is not rejected}\}$; get level $1 - \alpha$ confidence set for ϕ.
- **Example**: Usual t test gives rise in this way to the usual t confidence intervals
$$\bar{X} \pm t_{n-1,\alpha/2} \frac{s}{\sqrt{n}}.$$

Confidence sets from Pivots

- **Definition**: A **pivot** (pivotal quantity) is a function $g(\theta, X)$ whose distribution is the same for all θ.
- Note θ in pivot is same θ as being used to calculate distribution of $g(\theta, X)$.
- Using pivots to generate confidence sets:
- Pick a set A in space of possible values for g.
- Let $\beta = P_\theta(g(\theta, X) \in A)$; since g is pivotal β is the same for all θ.
- Given data X solve the relation

$$g(\theta, X) \in A$$

to get

$$\theta \in C(X, A) .$$

Example: Normal variance interval

- Note $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$ is pivot in $N(\mu, \sigma^2)$ model.
- Given $\beta = 1 - \alpha$ consider the two points

$$\chi^2_{n-1,1-\alpha/2} \text{ and } \chi^2_{n-1,\alpha/2}.$$

- Then

$$P(\chi^2_{n-1,1-\alpha/2} \le (n-1)s^2/\sigma^2 \le \chi^2_{n-1,\alpha/2}) = \beta$$

  for all μ, σ.

- Solve this relation:

$$P(\frac{(n-1)^{1/2}s}{\chi_{n-1,\alpha/2}} \le \sigma \le \frac{(n-1)^{1/2}s}{\chi_{n-1,1-\alpha/2}}) = \beta$$

  so interval

$$\left[\frac{(n-1)^{1/2}s}{\chi_{n-1,\alpha/2}}, \frac{(n-1)^{1/2}s}{\chi_{n-1,1-\alpha/2}}\right]$$

  is a level $1-\alpha$ confidence interval.

Other intervals

- In the same model we also have
$$P(\chi^2_{n-1,1-\alpha} \le (n-1)s^2/\sigma^2) = \beta$$
which can be solved to get
$$P(\sigma \le \frac{(n-1)^{1/2}s}{\chi_{n-1,1-\alpha}}) = \beta$$
- This gives a level $1-\alpha$ interval
$$(0, (n-1)^{1/2}s/\chi_{n-1,1-\alpha}) .$$
- Right hand end of interval usually called *confidence upper bound*.
- In general the interval from
$$(n-1)^{1/2}s/\chi_{n-1,\alpha_1} \text{ to } (n-1)^{1/2}s/\chi_{n-1,1-\alpha_2}$$
has level $\beta = 1-\alpha_1-\alpha_2$.
- For fixed β can minimize length of interval numerically — rarely used.
- See homework for an example.