0.0.1 Comparing modes of convergence

Think about a sequence X_n and a possible limit X:

- X_n converges in distribution to X depends *only* on the marginal distributions of individual X_n and X.
- Convergence in probability and pth mean depends only on the sequence of bivariate joint distributions of (X_n, X).
- Convergence almost surely depends on the joint distribution of all the variables: $X_1, X_2, \ldots, X$.

All these convergences depend on scaling!

- In an iid sequence $\bar{X}_n$ converges in all senses to $\mu = \mathrm{E}(X_1)$ (for pth mean add the hypothesis that $\mathrm{E}(|X_1|^p) < \infty$. Y.
- In addition $\sqrt{n}(\bar{X}_n - \mu)$ converges in distribution to a normal random variable if $\mathrm{Var}(X_1) < \infty$.
- But not in any of the other senses of convergence.

Relation between modes of convergence

- If X_n converges to X almost surely then X_n converges to X in probability.
- If X_n converges to X in probability then X_n converges to X in distribution.
- If X_n converges to X in pth mean for some $p > 0$ then X_n converges to X in probability.
- If X_n converges to X in probability and the sequence is *uniformly pth power integrable* then X_n converges to X in pth mean.
- **Definition**: Uniformly pth power integrable means
$$\lim_{M \to \infty} \sup\{\mathrm{E}(|X_n|^p 1(|X_n| > M)) = 0.$$
- Most easily checked by: $\exists \delta > 0$ such that
$$\sup\{\mathrm{E}(|X_n|^{p+\delta}) < \infty.$$

Some examples

- We generate observation from the exponential and Cauchy distribution, that is, generate $X_1, X_2, \cdots$ independently from these distributions.
- Generation is done in batches of 100.
- Generate a total of 10000 batches.
- Plots:
  - Sample mean $\bar{X}_n = \sum_{i=1}^{n} X_n/n$ against n.