

STAT 830

Convergence of RVs

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Purposes of These Notes

- Distinguish convergence in distribution from other modes of convergence.
- Describe which modes of convergence imply which others.

- Think about a sequence X_n and a possible limit X :
 - ▶ X_n converges in distribution to X depends *only* on marginal distributions of individual X_n and X .
 - ▶ Convergence in probability and p th mean depends only on sequence of bivariate joint distributions of (X_n, X) .
 - ▶ Convergence almost surely depends on joint distribution of all the variables: $X_1, X_2, \dots, X$.
- All depend on scaling!
- In an iid sequence $\bar{X}_n$ converges in all senses to $\mu = E(X_1)$ (for p th mean add the hypothesis that $E(|X_1|^p) < \infty$).
- In addition $\sqrt{n}(\bar{X}_n - \mu)$ converges in distribution to a normal random variable if $\text{Var}(X_1) < \infty$.
- But not in any of the other senses of convergence.

Relation between modes of convergence

- If X_n converges to X almost surely then X_n converges to X in probability.
- If X_n converges to X in probability then X_n converges to X in distribution.
- If X_n converges to X in p th mean for some $p > 0$ then X_n converges to X in probability.
- If X_n converges to X in probability and the sequence is *uniformly p th power integrable* then X_n converges to X in p th mean.
- **Definition:** Uniformly p th power integrable means

$$\lim_{M \rightarrow \infty} \sup \{E(|X_n|^p 1(|X_n| > M))\} = 0.$$

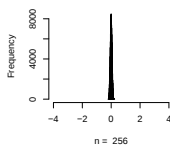
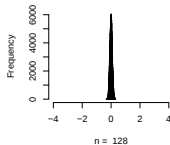
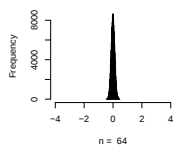
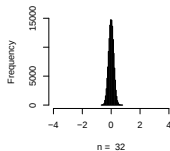
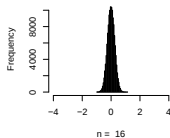
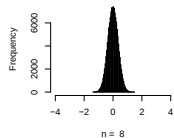
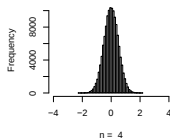
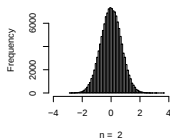
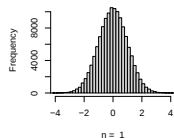
- Most easily checked by: $\exists \delta > 0$ such that

$$\sup \{E(|X_n|^{p+\delta})\} < \infty.$$

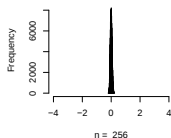
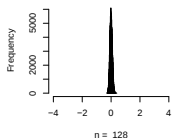
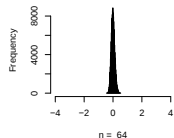
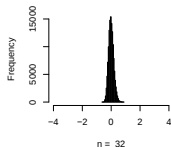
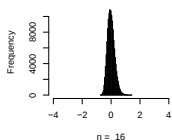
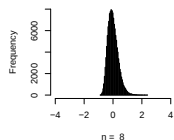
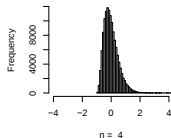
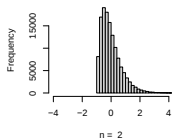
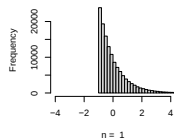
Some examples

- We generate observation from the exponential and Cauchy distribution, that is, generate $X_1, X_2, \dots$ independently from these distributions.
- Generate $n = 2^{17}$ observations.
- Plots:
 - ▶ Sample mean $\bar{X}_n = \sum_{i=1}^n X_i/n$ against n . (In class.)
 - ▶ Histogram of 2^{17-j} values of averages of 2^j data points for $j = 1, \dots, 9$.

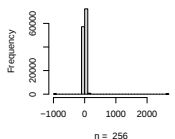
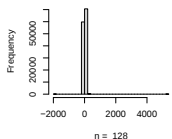
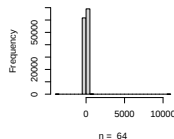
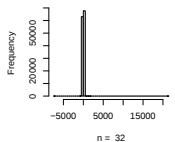
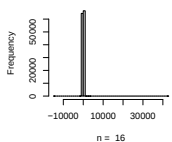
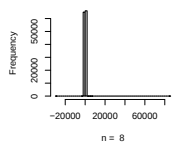
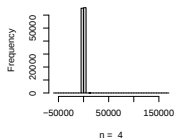
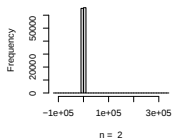
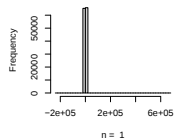
Normal samples – histograms



Exponential samples – histograms



Cauchy samples – histograms



Cauchy samples zoomed in – histograms

