

STAT 830

Independence and Conditioning

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Purposes of These Notes

- Define independent events and random variables.
- Give conditions for independence.
- Define conditional probability, conditional distribution.
- State Bayes Theorem in various forms.



Def'n: Events A and B are independent if

$$P(AB) = P(A)P(B).$$

(Notation: AB is the event that both A and B happen, also written $A \cap B$.)

Def'n: $A_i, i = 1, \dots, p$ are **independent** if

$$P(A_{i_1} \cdots A_{i_r}) = \prod_{j=1}^r P(A_{i_j})$$

for any $1 \leq i_1 < \cdots < i_r \leq p$.

Example: $p = 3$

$$P(A_1 A_2 A_3) = P(A_1)P(A_2)P(A_3)$$

$$P(A_1 A_2) = P(A_1)P(A_2)$$

$$P(A_1 A_3) = P(A_1)P(A_3)$$

$$P(A_2 A_3) = P(A_2)P(A_3)$$

All these equations needed for independence!



Counterexample

- Pairwise independence is not independence.
- Toss a coin twice.

$A_1 = \{\text{first toss is a Head}\}$

$A_2 = \{\text{second toss is a Head}\}$

$A_3 = \{\text{first toss and second toss different}\}$

- Then $P(A_i) = 1/2$ for each i and for $i \neq j$

$$P(A_i \cap A_j) = \frac{1}{4}$$

but

$$P(A_1 \cap A_2 \cap A_3) = 0 \neq P(A_1)P(A_2)P(A_3).$$



Def'n: X and Y are **independent** if

$$P(X \in A; Y \in B) = P(X \in A)P(Y \in B)$$

for all A and B .

Notation: Write $X \perp\!\!\!\perp Y$.

Def'n: Rvs $X_1, \dots, X_p$ **independent:**

$$P(X_1 \in A_1, \dots, X_p \in A_p) = \prod P(X_i \in A_i)$$

for any $A_1, \dots, A_p$.



Theorem

- ❶ If X and Y are independent then for all x, y

$$F_{X,Y}(x, y) = F_X(x)F_Y(y).$$

- ❷ If X and Y are independent with joint density $f_{X,Y}(x, y)$ then X and Y have densities f_X and f_Y , and

$$f_{X,Y}(x, y) = f_X(x)f_Y(y).$$

- ❸ If X and Y independent with marginal densities f_X and f_Y then (X, Y) has joint density

$$f_{X,Y}(x, y) = f_X(x)f_Y(y).$$



Theorem Continued

Theorem (Theorem Continued)

- 4 If $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ for **all** x,y then X and Y are independent.
- 5 If (X,Y) has density $f(x,y)$ and there exist $g(x)$ and $h(y)$ st $f(x,y) = g(x)h(y)$ for (almost) **all** (x,y) then X and Y are independent with densities given by

$$f_X(x) = g(x) / \int_{-\infty}^{\infty} g(u) du$$

$$f_Y(y) = h(y) / \int_{-\infty}^{\infty} h(u) du.$$

- 6 An analogous assertion to the previous holds in the discrete case.



Proof of First Assertion

- Since X and Y are independent the events $X \leq x$ and $Y \leq y$ are independent
- So

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y).$$



Proof of second assertion

- Suppose X and Y real valued.
- Asst 2: existence of $f_{X,Y}$ implies that of f_X and f_Y (marginal density formula).
- Then for any sets A and B

$$\begin{aligned}P(X \in A, Y \in B) &= \int_A \int_B f_{X,Y}(x, y) dy dx \\P(X \in A)P(Y \in B) &= \int_A f_X(x) dx \int_B f_Y(y) dy \\&= \int_A \int_B f_X(x) f_Y(y) dy dx.\end{aligned}$$

- Since $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$

$$\int_A \int_B [f_{X,Y}(x, y) - f_X(x)f_Y(y)] dy dx = 0.$$

Measure theory shows quantity in $[]$ is 0 for almost every pair (x, y)



Proof of third assertion

- For any A and B we have

$$\begin{aligned}P(X \in A, Y \in B) &= P(X \in A)P(Y \in B) \\&= \int_A f_X(x)dx \int_B f_Y(y)dy \\&= \int_A \int_B f_X(x)f_Y(y)dydx.\end{aligned}$$

If we **define** $g(x, y) = f_X(x)f_Y(y)$ then we have proved that for $C = A \times B$

$$P((X, Y) \in C) = \int_C g(x, y)dydx.$$

- To prove that g is $f_{X,Y}$ prove this integral formula is valid for arbitrary Borel set C , not just rectangle $A \times B$.
- Use *monotone class* argument. Study closure properties collection of sets C for which identity holds.



Proof of fourth and fifth assertions

- For fourth assertion another monotone class argument.
- For fifth assertion:

$$\begin{aligned}P(X \in A, Y \in B) &= \int_A \int_B g(x)h(y)dydx \\ &= \int_A g(x)dx \int_B h(y)dy.\end{aligned}$$

Take $B = R^1$ to see that

$$P(X \in A) = c_1 \int_A g(x)dx$$

where $c_1 = \int h(y)dy$.

- So $c_1 g$ is the density of X . Since $\int \int f_{X,Y}(xy)dxdy = 1$ we see that $\int g(x)dx \int h(y)dy = 1$ so that $c_1 = 1/\int g(x)dx$.
- Similar argument for Y .



Inheritance of transformations

Theorem

If $X_1, \dots, X_p$ are independent and $Y_i = g_i(X_i)$ then $Y_1, \dots, Y_p$ are independent. Moreover, $(X_1, \dots, X_q)$ and $(X_{q+1}, \dots, X_p)$ are independent. (In fact everything you would expect to hold does.)



Def'n: $P(A|B) = P(AB)/P(B)$ if $P(B) \neq 0$.

Def'n: For discrete X and Y the conditional probability mass function of Y given X is

$$\begin{aligned}f_{Y|X}(y|x) &= P(Y = y|X = x) \\&= f_{X,Y}(x, y)/f_X(x) \\&= f_{X,Y}(x, y)/\sum_t f_{X,Y}(x, t)\end{aligned}$$



- For absolutely continuous X $P(X = x) = 0$ for all x .
- What is $P(A|X = x)$ or $f_{Y|X}(y|x)$?
- Solution: use limit

$$P(A|X = x) = \lim_{\delta x \rightarrow 0} P(A|x \leq X \leq x + \delta x)$$

- If, e.g., X, Y have joint density $f_{X,Y}$ then with $A = \{Y \leq y\}$ we have

$$\begin{aligned} P(A|x \leq X \leq x + \delta x) &= \frac{P(A \cap \{x \leq X \leq x + \delta x\})}{P(x \leq X \leq x + \delta x)} \\ &= \frac{\int_{-\infty}^y \int_x^{x+\delta x} f_{X,Y}(u, v) du dv}{\int_x^{x+\delta x} f_X(u) du} \end{aligned}$$

- Divide top, bottom by δx ; let $\delta x \rightarrow 0$.
- Denom converges to $f_X(x)$; numerator converges to

$$\int_{-\infty}^y f_{X,Y}(x, v) dv$$



Continuous case continued

- Define conditional cdf of Y given $X = x$:

$$P(Y \leq y | X = x) = \frac{\int_{-\infty}^y f_{X,Y}(x, v) dv}{f_X(x)}$$

- Differentiate wrt y to get def'n of conditional density of Y given $X = x$:

$$f_{Y|X}(y|x) = f_{X,Y}(x, y) / f_X(x);$$

in words “conditional = joint/marginal”.



- From $P(AB) = P(A|B)P(B) = P(B|A)P(A)$ get

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

- Statistical description of difference between $B \implies A$ and $A \implies B$.
- Density formulation

$$f_{X|Y} = \frac{f_{Y|X}f_X}{f_Y}$$

- Bayesians like to write

$$(x|y) = (y|x)(x)/(y)$$

with the parentheses indicating densities and the letters indicating variables.



Generalizations

- More general formulas arise like

$$P(ABCD) = P(A|BCD)P(B|CD)P(C|D)P(D)$$

- Also: if $A_1, \dots, A_k$ *mutually exclusive and exhaustive* then

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{\sum_i P(B|A_i)P(A_i)}$$

- *Mutually exclusive* means pairwise disjoint and *exhaustive* means

$$\cup_1^k A_i = \Omega.$$

- The density formula is really analogous since integrals are limits of sums

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{f_{Y|X}(y|x)f_X(x)}{\int_u f_{XY}(u, y)du}.$$

