STAT 830
Likelihood Ratio Tests

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Purposes of These Notes

- Describe likelihood ratio tests
- Discuss large sample χ^2 approximation.
- Discuss level and power
- Discuss quadratic forms in MVN vectors.

Likelihood Ratio Tests

- For general composite hypotheses optimality theory is not usually successful in producing an optimal test.
- Instead: **heuristics**
- Consider likelihood ratio

$$\frac{f_{\theta_1}(X)}{f_{\theta_0}(X)}$$

- Choose $\theta_1 \in \Theta_1$ and $\theta_0 \in \Theta_0$
- Estimates of θ assuming respectively the alternative or null hypothesis is true.
- Simplest: each θ_i MLE, maximized only over Θ_i.

Example 1: $N(\mu, 1)$

- Test $\mu \le 0$ against $\mu > 0$. (Remember UMP test.)
- Log likelihood is

$$-n(\bar{X} - \mu)^2/2$$

- If $\bar{X} > 0$ then global maximum in Θ_1 at $\bar{X}$.
- If $\bar{X} \le 0$ global maximum in Θ_1 at 0.
- Thus $\hat{\mu}_1$ which Max $\ell(\mu)$ subject to $\mu > 0$ at $\hat{\mu}_1 = \bar{X}1(\bar{X} > 0)$.
- Similarly, $\hat{\mu}_0$ is $\bar{X}$ if $\bar{X} \le 0$ and 0 if $\bar{X} > 0$.

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One sided normal mean cont'd

- Hence
$$\frac{f_{\hat\theta_1}(X)}{f_{\hat\theta_0}(X)} = \exp\{\ell(\hat\mu_1) - \ell(\hat\mu_0)\} = \exp\{n\bar X|\bar X|/2\}$$
- Monotone increasing ftn of $\bar X$: rejection region $\bar X > K$.
- To get level α reject if $n^{1/2}\bar X > z_\alpha$.
- Notice simpler statistic is *log likelihood ratio*
$$\lambda \equiv 2\log\left(\frac{f_{\hat\mu_1}(X)}{f_{\hat\mu_0}(X)}\right) = n\bar X|\bar X|$$

Example 2: $H_o : \mu = 0$ in $N(\mu, 1)$

- Value of $\hat{\mu}_0$ is 0
- Maximum of log-likelihood over alternative $\mu \neq 0$ occurs at $\bar{X}$.
- This gives
$$\lambda = n\bar{X}^2$$
which has a χ_1^2 distribution.
- This test leads to the rejection region $\lambda > (z_{\alpha/2})^2$ which is the usual (UMPU) z-test.

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Example 3: $N(\mu, \sigma^2)$ model, $\mu = 0$ against $\mu \neq 0$

- Must find two estimates of μ, σ^2.
- Maximum likelihood over alternative occurs at global mle $\bar{X}, \hat{\sigma}^2$.
- We find
$$\ell(\hat{\mu}, \hat{\sigma}^2) = -n/2 - n\log(\hat{\sigma})$$
- Maximize ℓ over null hypothesis.
- Recall
$$\ell(\mu, \sigma) = -\frac{1}{2\sigma^2}\sum(X_i - \mu)^2 - n\log(\sigma)$$
- On null $\mu = 0$ so find $\hat{\sigma}_0$ by maximizing
$$\ell(0, \sigma) = -\frac{1}{2\sigma^2}\sum X_i^2 - n\log(\sigma)$$

LRT – general description

- This leads to
$$\hat\sigma_0^2 = \sum X_i^2/n$$
and
$$\ell(0,\hat\sigma_0) = -n/2 - n\log(\hat\sigma_0)$$
- This gives
$$\lambda = -n\log(\hat\sigma^2/\hat\sigma_0^2)$$
- Since
$$\frac{\hat\sigma^2}{\hat\sigma_0^2} = \frac{\sum(X_i-\bar X)^2}{\sum(X_i-\bar X)^2 + n\bar X^2}$$
we can write
$$\lambda = n\log(1+t^2/(n-1))$$
where
$$t = \frac{n^{1/2}\bar X}{s}$$
is the usual t statistic.
- LRT rejects for large values of $|t|$ — the usual test.

Large sample behaviour

- Notice that if n is large we have

$$\lambda \approx n[t^2/(n-1) + O_P(n^{-2})] \approx t^2\,.$$

- Since t statistic is approximately standard normal if n large we see

$$\lambda = 2[\ell(\hat\theta_1) - \ell(\hat\theta_0)]$$

has nearly a χ^2_1 distribution.

LRT – general description

- General phenomenon when null hypothesis has form $\phi = 0$.
- **Warning**: null should not be on edge of Θ.
- Suppose vector θ of $p + q$ parameters partitioned into $\theta = (\phi, \gamma)$
- ϕ a vector of p pars and γ a vector of q pars.
- To test $\phi = \phi_0$ we find two mles of θ.
- First: global mle $\hat\theta = (\hat\phi, \hat\gamma)$ maximizes likelihood over $\Theta_1 = \{\theta : \phi \neq \phi_0\}$ (typically $P_\theta(\hat\phi = \phi_0) = 0$).

LRT – general description

- Maximize likelihood over null hypothesis, that is find $\hat{\theta}_0 = (\phi_0, \hat{\gamma}_0)$ to maximize

$$\ell(\phi_0, \gamma)$$

- The log-likelihood ratio statistic is

$$2[\ell(\hat{\theta}) - \ell(\hat{\theta}_0)]$$

- Suppose true value of θ is ϕ_0, γ_0 (so null hypothesis is true).
- Score function is a vector of length $p + q$ and can be partitioned as $U = (U_\phi, U_\gamma)$.
- The Fisher information matrix can be partitioned as

$$\begin{bmatrix} \mathcal{I}_{\phi\phi} & \mathcal{I}_{\phi\gamma} \\ \mathcal{I}_{\gamma\phi} & \mathcal{I}_{\gamma\gamma} \end{bmatrix}.$$

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Large sample theory for LRT

- According to our large sample theory for the mle we have

$$\hat\theta \approx \theta + \mathcal{I}^{-1} U$$

  and

$$\hat\gamma_0 \approx \gamma_0 + \mathcal{I}_{\gamma\gamma}^{-1} U_\gamma$$

- Two term Taylor expansions of both $\ell(\hat\theta)$ around θ_0 gives

$$\ell(\hat\theta) \approx \ell(\theta_0) + U^t\mathcal{I}^{-1}U + \frac{1}{2}U^t\mathcal{I}^{-1}V(\theta)\mathcal{I}^{-1}U$$

  where V is the second derivative matrix of ℓ.

Large sample theory for LRT

- Remember that $V \approx -\mathcal{I}$ and you get

$$2[\ell(\hat\theta) - \ell(\theta_0)] \approx U^t \mathcal{I}^{-1} U\,.$$

- A similar expansion for $\hat\theta_0$ gives

$$2[\ell(\hat\theta_0) - \ell(\theta_0)] \approx U_\gamma^t \mathcal{I}_{\gamma\gamma}^{-1} U_\gamma\,.$$

- Subtract two expansions to write $2[\ell(\hat\theta) - \ell(\hat\theta_0)]$ in the approximate form

$$U^t M U$$

  for a suitable matrix M.
- Use distribution of $X^t M X$ where X is $MVN(0, \Sigma)$.

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The theorem: large sample theory of LRT

The ideas above lead to a proof of the following theorem.

Theorem

The log-likelihood ratio statistic

$$\lambda = 2[\ell(\hat\theta) - \ell(\hat\theta_0)]$$

has, under the null hypothesis, approximately a χ^2_p *distribution.*

Warning: requires regularity conditions including $\theta_0 = (\phi_0, \psi_0)$ is in the interior of Θ.

Quadratic forms and χ^2

In proving the main theorem we need some facts about quadratic forms.

Theorem

Suppose $X \sim MVN(0, \Sigma)$ with Σ non-singular and M is a symmetric matrix. If $\Sigma M \Sigma M \Sigma = \Sigma M \Sigma$ then $X^t M X$ has a χ^2_ν distribution with df $\nu = trace(M\Sigma)$. The condition simplifies to $M\Sigma M = M$

Proof

- We have $X = AZ$ where $AA^t = \Sigma$ and Z is standard multivariate normal.
- So $X^t MX = Z^t A^t MAZ$.
- Let $Q = A^t MA$.
- Since $AA^t = \Sigma$ condition in the theorem is

$$AQQA^t = AQA^t$$

- Since Σ is non-singular so is A.
- Multiply by A^{-1} on left and $(A^t)^{-1}$ on right; get $QQ = Q$.
- Jargon: Q is *idempotent*.

Proof

- Q is symmetric so $Q = P\Lambda P^t$ where
  - Λ is diagonal matrix
  - diagonal contains the eigenvalues of Q
  - P is orthogonal matrix: $P^\top P = Id$.
  - columns of P are corresponding orthonormal eigenvectors.
- So

$$Z^t QZ = (P^t Z)^t \Lambda (PZ) .$$

More proof

- $W = P^t Z$ is $MVN(0, P^t P = I)$; i.e. W is standard multivariate normal.
- Now
$$W^t \Lambda W = \sum \lambda_i W_i^2$$
- We have established that the general distribution of any quadratic form $X^t M X$ is a linear combination of χ^2 variables.
- Now go back to the condition $QQ = Q$.
- If λ is an eigenvalue of Q and $v \neq 0$ is a corresponding eigenvector then $QQv = Q(\lambda v) = \lambda Qv = \lambda^2 v$ but also $QQv = Qv = \lambda v$.
- Thus $\lambda(1 - \lambda)v = 0$.
- It follows that either $\lambda = 0$ or $\lambda = 1$.

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End of proof

- This means that the weights in the linear combination are all 1 or 0 and that $X^t M X$ has a χ^2 distribution with degrees of freedom, ν, equal to the number of λ_i which are equal to 1.
- This is the same as the sum of the λ_i so

$$\nu = trace(\Lambda)$$

- But

$$\begin{aligned} trace(M\Sigma) &= trace(MAA^t) \\ &= trace(A^t MA) \\ &= trace(Q) \\ &= trace(P\Lambda P^t) \\ &= trace(\Lambda P^t P) \\ &= trace(\Lambda) \end{aligned}$$

Application to LRT

- In the application Σ is $\mathcal{I}$ the Fisher information and $M = \mathcal{I}^{-1} - J$ where
$$J = \left[\begin{array}{cc} 0 & 0 \\ 0 & \mathcal{I}_{\gamma\gamma}^{-1} \end{array} \right]$$
- It is easy to check that $M\Sigma$ becomes
$$\left[\begin{array}{cc} I & 0 \\ -\mathcal{I}_{\gamma\phi}\mathcal{I}_{\phi\phi} & 0 \end{array} \right]$$
where I is a $p \times p$ identity matrix.
- It follows that $\Sigma M \Sigma M \Sigma = \Sigma M \Sigma$ and $trace(M\Sigma) = p$.