# STAT 870 Lecture 14

### Properties of Poisson Processes

1) If $N_1$ and $N_2$ are independent Poisson processes with rates $\lambda_1$ and $\lambda_2$, respectively, then $N = N_1 + N_2$ is a Poisson processes with rate $\lambda_1 + \lambda_2$.

2) Suppose $N$ is a Poisson process with rate $\lambda$. Suppose each point is marked with a label, say one of $L_1, \ldots, L_r$, independently of all other occurrences. Suppose $p_i$ is the probability that a given point receives label $L_i$. Let $N_i$ count the points with label $i$ (so that $N = N_1 + \cdots + N_r$). Then $N_1, \ldots, N_r$ are independent Poisson processes with rates $p_i\lambda$.

3) Suppose $U_1, U_2, \ldots$ independent rvs, each uniformly distributed on $[0, T]$. Suppose $M$ is a Poisson($\lambda T$) random variable independent of the $U$'s. Let
$$N(t) = \sum_1^M 1(U_i \le t)$$
Then $N$ is a Poisson process on $[0, T]$ with rate $\lambda$.

4) Suppose $N$ is a Poisson process with rate $\lambda$. Let $S_1 < S_2 < \cdots$ be the times at which points arrive Given $N(T) = n$, $S_1, \ldots, S_n$ have the same distribution as the order statistics of a sample of size $n$ from the uniform distribution on $[0, T]$.

5) Given $S_{n+1} = t$, $S_1, \ldots, S_n$ have the same distribution as the order statistics of a sample of size $n$ from the uniform distribution on $[0, T]$.

Indications of some proofs:

**1)** $N_1, \ldots, N_r$ independent Poisson processes rates $\lambda_i$, $N = \sum N_i$. Let $A_h$ be the event of 2 or more points in $N$ in the time interval $(t, t+h]$, $B_h$, the event of exactly one point in $N$ in the time interval $(t, t+h]$.

Let $A_{ih}$ and $B_{ih}$ be the corresponding events for $N_i$.

Let $H_t$ denote the history of the processes up to time $t$; we condition on $H_t$. Technically, $H_t$ is the $\sigma$-field generated by
$$\{N_i(s); 0 \le s \le t, i = 1, \ldots, r\}$$

We are given:

$$P(A_{ih}|H_t) = o(h)$$

and

$$P(B_{ih}|H_t) = \lambda_i h + o(h)\,.$$

Note that

$$A_h \subset \bigcup_{i=1}^{r} A_{ih} \cup \bigcup_{i \neq j} (B_{ih} \cap B_{jh})$$

Since

$$\begin{aligned} P(B_{ih} \cap B_{jh}|H_t) &= P(B_{ih}|H_t)P(B_{jh}|H_t) \\ &= (\lambda_i h + o(h))(\lambda_j h + o(h)) \\ &= O(h^2) \\ &= o(h) \end{aligned}$$

and

$$P(A_{ih}|H_t) = o(h)$$

we have checked one of the two infinitesimal conditions for a Poisson process.

Next let $C_h$ be the event of no points in $N$ in the time interval $(t, t+h]$ and $C_{ih}$ the same for $N_i$. Then

$$\begin{aligned} P(C_h|H_t) &= P(\cap C_{ih}|H_t) \\ &= \prod P(C_{ih}|H_t) \\ &= \prod (1 - \lambda_i h + o(h)) \\ &= 1 - (\sum \lambda_i)h + o(h) \end{aligned}$$

shows

$$\begin{aligned} P(B_h|H_t) &= 1 - P(C_h|H_t) - P(A_h|H_t) \\ &= (\sum \lambda_i)h + o(h) \end{aligned}$$

Hence $N$ is a Poisson process with rate $\sum \lambda_i$.

**2)** The infinitesimal approach used for 1 can do part of this. See text for rest. Events defined as in **1)**: The event $B_{ih}$ that there is one point in $N_i$ in $(t, t+h]$ is the event, $B_h$ that there is exactly one point in any of the $r$

processes together with a subset of $A_h$ where there are two or more points in $N$ in $(t, t+h]$ but exactly one is labeled $i$. Since $P(A_h|H_t) = o(h)$

$$\begin{aligned} P(B_{ih}|H_t) &= p_i P(B_h|H_t) + o(h) \\ &= p_i(\lambda h + o(h)) + o(h) \\ &= p_i \lambda h + o(h) \end{aligned}$$

Similarly, $A_{ih}$ is a subset of $A_h$ so

$$P(A_{ih}|H_t) = o(h)$$

This shows each $N_i$ is Poisson with rate $\lambda p_i$. To get independence requires more work; see the homework for an easier algebraic method.

**3)** Fix $s < t$. Let $N(s,t)$ be the number of points in $(s,t]$. Given $N = n$ the conditional distribution of $N(s,t)$ is Binomial$(n,p)$ with $p = (s-t)/T$. So

$$\begin{aligned} &P(N(s,t) = k) \\ &\quad = \sum_{n=k}^{\infty} P(N(s,t) = k, N = n) \\ &\quad = \sum_{n=k}^{\infty} P(N(s,t) = k | N = n) P(N = n) \\ &\quad = \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \frac{(\lambda T)^n}{n!} e^{-\lambda T} \\ &\quad = \frac{e^{-\lambda T}}{k!} (\lambda T p)^k \sum_{n=k}^{\infty} (1-p)^{n-k} (\lambda T)^{n-k} \\ &\quad = \frac{e^{-\lambda T}}{k!} (\lambda T p)^k \sum_{m=0}^{\infty} (1-p)^m (\lambda T)^m \\ &\quad = \frac{e^{-\lambda T}}{k!} (\lambda T p)^k e^{-\lambda T(1-p)} \\ &\quad = \frac{e^{-\lambda(s-t)} (\lambda(s-t))^k}{k!} \end{aligned}$$

**4)**: Fix $s_i, h_i$ for $i = 1, \ldots, n$ such that

$$0 < s_1 < s_1 + h_1 < s_2 < \cdots < s_n < s_n + h_n < T$$

Given $N(T) = n$ we compute the probability of the event

$$A = \bigcap_{i=1}^{n} \{s_i < S_i < s_i + h_i\}$$

Intersection of $A$,1 $N(T) = n$ is $(s_0 = h_0 = 0)$:

$$\begin{aligned} B \equiv & \\ & \bigcap_{i=1}^{n} \{N(s_{i-1} + h_{i-1}, s_i] = 0, N(s_i, s_i + h_i] = 1\} \\ & \qquad \cap \{N(s_n + h_n, T] = 0\} \end{aligned}$$

whose probability is

$$\left(\prod \lambda h_i\right) e^{-\lambda T}$$

So

$$\begin{aligned} P(A|N(t) = n) &= \frac{P(A, N(T) = n)}{(N(T) = n)} \\ &= \frac{\lambda^n e^{-\lambda T} \prod h_i}{(\lambda T)^n e^{-\lambda T}/n!} \\ &= \frac{n! \prod h_i}{T^n} \end{aligned}$$

Divide by $\prod h_i$ and let all $h_i$ go to 0 to get joint density of $S_1, \ldots, S_n$ is

$$\frac{n!}{T^n} 1(0 < s_1 < \cdots < s_n < T)$$

which is the density of order statistics from a Uniform$[0, T]$ sample of size $n$.

**5)** Replace the event $S_{n+1} = T$ with $T < S_{n+1} < T + h$. With $A$ as before we want

$$\begin{aligned} & P(A|T < S_{n+1} < T + h) \\ & \qquad = \frac{P(B, N(T, T + h] \geq 1)}{P(T < S_{n+1} < T + h)} \end{aligned}$$

Note that $B$ is independent of $\{N(T, T + h] \geq 1\}$ and that we have already found the limit

$$\frac{P(B)}{\prod h_i} \to \lambda^n e^{-\lambda T}$$

We are left to compute the limit of

$$\frac{P(N(T,T+h] \ge 1)}{P(T < S_{n+1} < T+h)}$$

The denominator is

$$F_{S_{n+1}}(t+h) - F_{S_{n+1}}(t) = f_{S_{n+1}}(t)h + o(h)$$

Thus

$$\begin{aligned}\frac{P(N(T,T+h] \ge 1)}{P(T < S_{n+1} < T+h)} &= \frac{\lambda h + o(h)}{\frac{(\lambda T)^n}{n!}e^{-\lambda T}\lambda h + o(h)} \\ &\to \frac{n!}{(\lambda T)^n e^{-\lambda T}}\end{aligned}$$

This gives the conditional density of $S_1, \ldots, S_n$ given $S_{n+1} = T$ as in **4)**.

## Inhomogeneous Poisson Processes

The idea of hazard rate can be used to extend the notion of Poisson Process. Suppose $\lambda(t) \ge 0$ is a function of $t$. Suppose $N$ is a counting process such that

$$P(N(t+h) = k+1 | N(t) = k, H_t) = \lambda(t)h + o(h)$$

and

$$P(N(t+h) \ge k+2 | N(t) = k, H_t) = o(h)$$

Then $N$ has independent increments and $N(t+s) - N(t)$ has a Poisson distribution with mean

$$\int_t^{t+s} \lambda(u)du$$

If we put

$$\Lambda(t) = \int_0^t \lambda(u)du$$

then mean of $N(t+s) - N(T)$ is $\Lambda(t+s) - \Lambda(t)$.

Jargon: $\lambda$ is the **intensity** or **instantaneous intensity** and $\Lambda$ the **cumulative intensity**.

Can use the model with $\Lambda$ any non-decreasing right continuous function, possibly without a derivative. This allows ties.

## Space Time Poisson Processes

Suppose at each time $S_i$ of a Poisson Process, $N$, we have rv $Y_i$ with the $Y_i$ iid and independent of the Poisson process. Let $M$ be the counting process on $[0, \infty) \times \mathcal{Y}$ (where $\mathcal{Y}$ is the range space of the $Y$s) defined by

$$M(A) = \#\{(S_i, Y_i) \in A\}$$

Then $M$ is an inhomogeneous Poisson process with mean function $\mu$ a measure extending

$$\mu([a, b] \times C) = \lambda(b - a)P(Y \in C)$$

This means that each $M(A)$ has a Poisson distribution with mean $\mu(A)$ and if $A_1, \ldots, A_r$ are disjoint then $M(A_1), \ldots, M(A_r)$ are independent. The proof in general is a monotone class argument. The first step is: if $(a_i, b_i), i = 1, \ldots, r$ are disjoint intervals and $C_1, \ldots, C_s$ disjoint subsets of $\mathcal{Y}$ then the $rs$ rvs $M((a_i, b_i) \times C_j)$ are independent Poisson random variables. See the homework for proof of a special case.