Brownian Motion

For fair random walk Y_n = number of heads minus number of tails,

$$Y_n = U_1 + \cdots + U_n$$

where the U_i are independent and

$$P(U_i = 1) = P(U_i = -1) = \frac{1}{2}$$

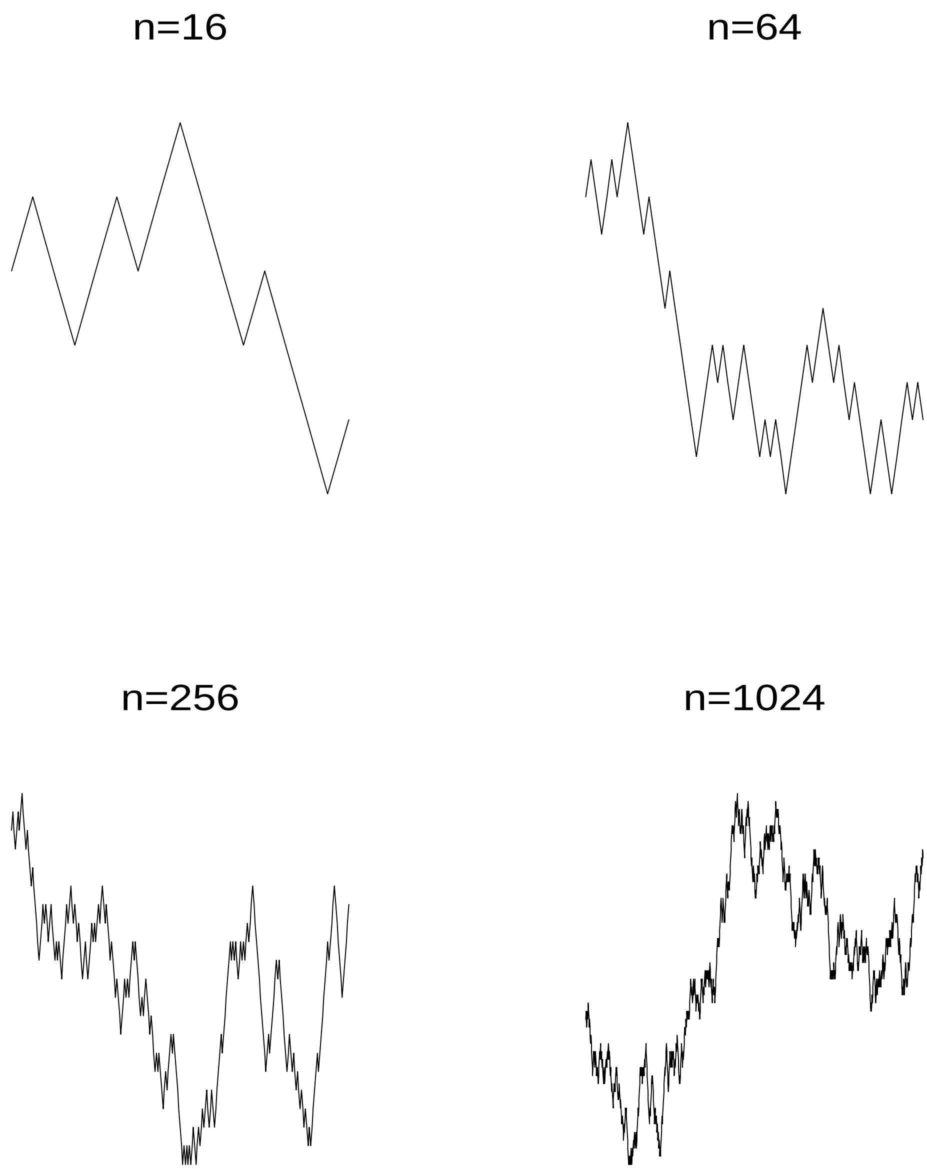
Notice:

$$\begin{aligned} \mathrm{E}(U_i) &= 0 \\ \mathrm{Var}(U_i) &= 1 \end{aligned}$$

Recall central limit theorem:

$$\frac{U_1 + \cdots + U_n}{\sqrt{n}} \Rightarrow N(0,1)$$

Now: rescale time axis so that n steps take 1 time unit and vertical axis so step size is $1/\sqrt{n}$.


n=16
n=64
n=256
n=1024

We now turn these pictures into a stochastic process:

For $\frac{k}{n} \le t < \frac{k+1}{n}$ we define

$$X_n(t) = \frac{U_1 + \cdots + U_k}{\sqrt{n}}$$

Notice:

$$\mathrm{E}(X_n(t)) = 0$$

and

$$\mathrm{Var}(X_n(t)) = \frac{k}{n}$$

As $n \to \infty$ with t fixed we see $k/n \to t$. Moreover:

$$\frac{U_1 + \cdots + U_k}{\sqrt{k}} = \sqrt{\frac{n}{k}} X_n(t)$$

converges to $N(0,1)$ by the central limit theorem. Thus

$$X_n(t) \Rightarrow N(0,t)$$

Also: $X_n(t+s) - X_n(t)$ is independent of $X_n(t)$ because the 2 rvs involve sums of different U_i.

Conclusions.

As $n \to \infty$ the processes X_n converge to a process X with the properties:

1. $X(t)$ has a $N(0, t)$ distribution.

2. X has independent increments: if

   $$0 = t_0 < t_1 < t_2 < \cdots < t_k$$

   then

   $$X(t_1) - X(t_0), \ldots, X(t_k) - X(t_{k-1})$$

   are independent .

3. The increments are **stationary**: for all s

   $$X(t+s) - X(s) \sim N(0, t)$$

4. $X(0) = 0$.

Def'n: Any process satisfying 1-4 above is a Brownian motion.

Properties of Brownian motion

- Suppose $t > s$. Then

$$\begin{aligned}\mathsf{E}(X(t)|X(s)) &= \mathsf{E}\left\{X(t) - X(s) + X(s)|X(s)\right\} \\ &= \mathsf{E}\left\{X(t) - X(s)|X(s)\right\} \\ &\quad + \mathsf{E}\left\{X(s)|X(s)\right\} \\ &= 0 + X(s) = X(s)\end{aligned}$$

Notice the use of independent increments and of $\mathsf{E}(Y|Y) = Y$.

- Again if $t > s$:

$$\begin{aligned}&\mathsf{Var}\left\{X(t)|X(s)\right\} \\ &\quad = \mathsf{Var}\left\{X(t) - X(s) + X(s)|X(s)\right\} \\ &\quad = \mathsf{Var}\left\{X(t) - X(s)|X(s)\right\} \\ &\quad = \mathsf{Var}\left\{X(t) - X(s)\right\} \\ &\quad = t - s\end{aligned}$$

Suppose $t < s$. Then $X(s) = X(t) + \{X(t) - X(s)\}$ is a sum of two independent normal variables. Do following calculation:

$X \sim N(0, \sigma^2)$, and $Y \sim N(0, \tau^2)$ independent. $Z = X + Y$.

Compute conditional distribution of X given Z:

$$\begin{aligned} f_{X|Z}(x|z) &= \frac{f_{X,Z}(x,z)}{f_Z(z)} \\ &= \frac{f_{X,Y}(x, z-x)}{f_Z(z)} \\ &= \frac{f_X(x) f_Y(z-x)}{f_Z(z)} \end{aligned}$$

Now Z is $N(0, \gamma^2)$ where $\gamma^2 = \sigma^2 + \tau^2$ so

$$\begin{aligned} f_{X|Z}(x|z) &= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)} \frac{1}{\tau\sqrt{2\pi}} e^{-(z-x)^2/(2\tau^2)}}{\frac{1}{\gamma\sqrt{2\pi}} e^{-z^2/(2\gamma^2)}} \\ &= \frac{\gamma}{\tau\sigma\sqrt{2\pi}} \exp\{-(x-a)^2/(2b^2)\} \end{aligned}$$

for suitable choices of a and b. To find them compare coefficients of x^2, x and 1.

Coefficient of x^2:

$$\frac{1}{b^2} = \frac{1}{\sigma^2} + \frac{1}{\tau^2}$$

so $b = \tau\sigma/\gamma$.

Coefficient of x:

$$\frac{a}{b^2} = \frac{z}{\tau^2}$$

so that

$$a = b^2 z/\tau^2 = \frac{\sigma^2}{\sigma^2+\tau^2} z$$

Finally you should check that

$$\frac{a^2}{b^2} = \frac{z^2}{\tau^2} - \frac{z^2}{\gamma^2}$$

to make sure the coefficients of 1 work out as well.

Conclusion: given $Z = z$ the conditional distribution of X is $N(a, b^2)$ with a and b as above.

Application to Brownian motion:

- For $t < s$ let X be $X(t)$ and Y be $X(s) - X(t)$ so $Z = X + Y = X(s)$. Then $\sigma^2 = t$, $\tau^2 = s - t$ and $\gamma^2 = s$. Thus

  $$b^2 = \frac{(s-t)t}{s}$$

  and

  $$a = \frac{t}{s} X(s)$$

  SO:

  $$\mathsf{E}(X(t)|X(s)) = \frac{t}{s} X(s)$$

  and

  $$\mathsf{Var}(X(t)|X(s)) = \frac{(s-t)t}{s}$$

The Reflection Principle

Tossing a fair coin:

HTHHHTHTHHTHHHTTHTH	5 more heads than tails
THTTTHTHTTHTTTHHTHT	5 more tails than heads

Both sequences have the same probability.

So: for random walk starting at stopping time:

Any sequence with k more heads than tails in next m tosses is matched to sequence with k more tails than heads. Both sequences have same prob.

Suppose Y_n is a fair ($p = 1/2$) random walk. Define

$$M_n = \max\{Y_k, 0 \le k \le n\}$$

Compute $P(M_n \ge x)$? Trick: Compute

$$P(M_n \ge x, Y_n = y)$$

First: if $y \ge x$ then

$$\{M_n \ge x, Y_n = y\} = \{Y_n = y\}$$

Second: if $M_n \ge x$ then

$$T \equiv \min\{k : Y_k = x\} \le n$$

Fix $y < x$. Consider a sequence of H's and T's which leads to say $T = k$ and $Y_n = y$.

Switch the results of tosses $k+1$ to n to get a sequence of H's and T's which has $T = k$ and $Y_n = x + (x - y) = 2x - y > x$. This proves

$$P(T = k, Y_n = y) = P(T = k, Y_n = 2x - y)$$

This is true for each k so

$$P(M_n \geq x, Y_n = y) = P(M_n \geq x, Y_n = 2x - y)$$
$$= P(Y_n = 2x - y)$$

Finally, sum over all y to get

$$P(M_n \geq x) = \sum_{y \geq x} P(Y_n = y)$$
$$+ \sum_{y < x} P(Y_n = 2x - y)$$

Make the substitution $k = 2x - y$ in the second sum to get

$$P(M_n \geq x) = \sum_{y \geq x} P(Y_n = y)$$
$$+ \sum_{k > x} P(Y_n = k)$$
$$= 2 \sum_{k > x} P(Y_n = k) + P(Y_n = x)$$

Brownian motion version:

$$M_t = \max\{X(s); 0 \le s \le t\}$$

$$T_x = \min\{s : X(s) = x\}$$

(called hitting time for level x). Then

$$\{T_x \le t\} = \{M_t \ge x\}$$

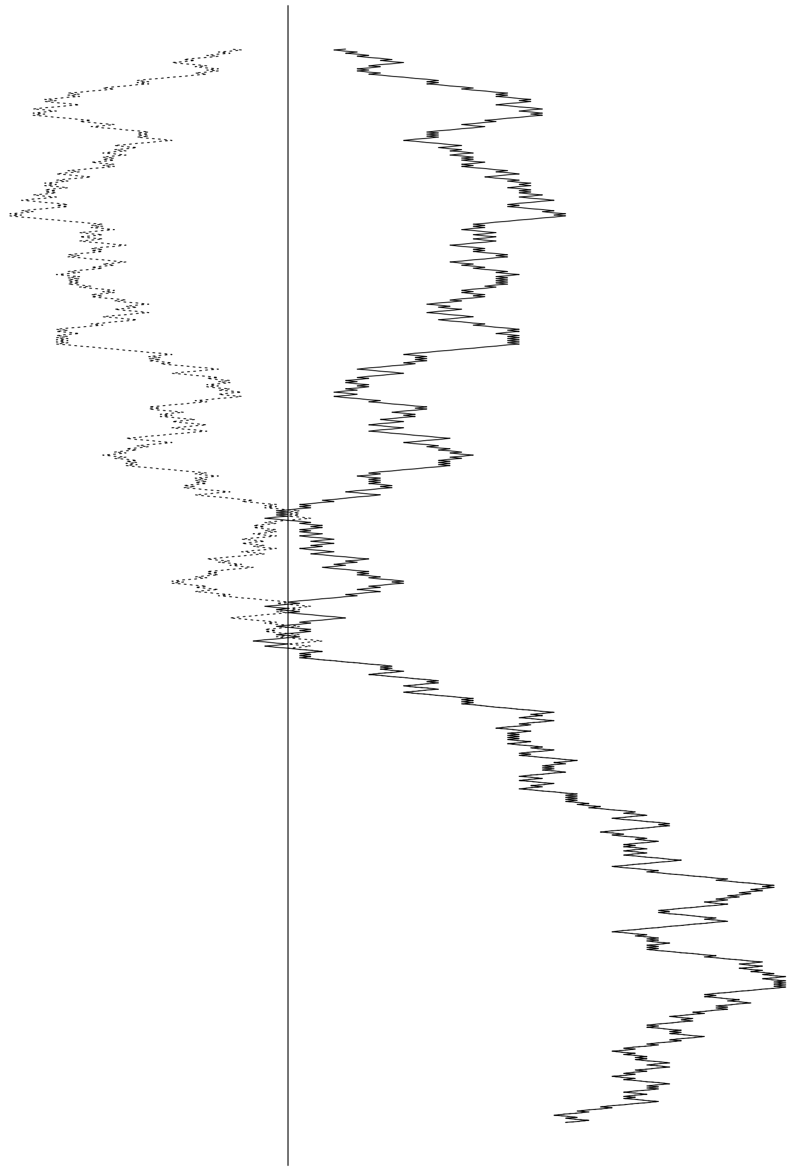
Any path with $T_x = s < t$ and $X(t) = y < x$ is matched to an equally likely path with $T_x = s < t$ and $X(t) = 2x - y > x$.

So for $y > x$

$$P(M_t \ge x, X(t) > y) = P(X(t) > y)$$

while for $y < x$

$$P(M_t \ge x, X(t) < y) = P(X(t) > 2x - y)$$



Let $y \to x$ to get

$$\begin{aligned} P(M_t \geq x, X(t) > x) &= P(M_t \geq x, X(t) < x) \\ &= P(X(t) > x) \end{aligned}$$

Adding these together gives

$$\begin{aligned} P(M_t > x) &= 2P(X(t) > x) \\ &= 2P(N(0,1) > x/\sqrt{t}) \end{aligned}$$

Hence M_t has the distribution of $|N(0,t)|$.

On the other hand in view of

$$\{T_x \le t\} = \{M_t \ge x\}$$

the density of T_x is

$$\frac{d}{dt} 2P(N(0,1) > x/\sqrt{t})$$

Use the chain rule to compute this. First

$$\frac{d}{dy} P(N(0,1) > y) = -\phi(y)$$

where ϕ is the standard normal density

$$\phi(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

because $P(N(0,1) > y)$ is 1 minus the standard normal cdf.

So

$$\frac{d}{dt} 2P(N(0,1) > x/\sqrt{t})$$
$$= -2\phi(x/\sqrt{t}) \frac{d}{dt}(x/\sqrt{t})$$
$$= \frac{x}{\sqrt{2\pi} t^{3/2}} \exp\{-x^2/(2t)\}$$

This density is called the **Inverse Gaussian** density. T_x is called a **first passage time**

NOTE: the preceding is a density when viewed as a function of the variable t.

Martingales

A stochastic process $M(t)$ indexed by either a discrete or continuous time parameter t is a **martingale** if:

$$\mathrm{E}\{M(t)|M(u); 0 \le u \le s\} = M(s)$$

whenever $s < t$.

Examples

- A fair random walk is a martingale.

- If $N(t)$ is a Poisson Process with rate λ then $N(t) - \lambda t$ is a martingale.

- Standard Brownian motion (defined above) is a martingale.

Note: Brownian motion with drift is a process of the form

$$X(t) = \sigma B(t) + \mu t$$

where B is **standard** Brownian motion, introduced earlier. X is a martingale if $\mu = 0$. We call μ the **drift**

- If $X(t)$ is a Brownian motion with drift then

$$Y(t) = e^{X(t)}$$

is a geometric Brownian motion. For suitable μ and σ we can make $Y(t)$ a martingale.

- If a gambler makes a sequence of fair bets and M_n is the amount of money s/he has after n bets then M_n is a martingale – even if the bets made depend on the outcomes of previous bets, that is, even if the gambler plays a strategy.

Some evidence for some of the above:

Random walk: $U_1, U_2, \ldots$ iid with

$$P(U_i = 1) = P(U_i = -1) = 1/2$$

and $Y_k = U_1 + \cdots + U_k$ with $Y_0 = 0$. Then

$$\begin{aligned}
\mathrm{E}(Y_n | Y_0, \ldots, Y_k) & \\
&= \mathrm{E}(Y_n - Y_k + Y_k | Y_0, \ldots, Y_k) \\
&= \mathrm{E}(Y_n - Y_k | Y_0, \ldots, Y_k) + Y_k \\
&= \sum_{k+1}^{n} \mathrm{E}(U_j | U_1, \ldots, U_k) + Y_k \\
&= \sum_{k+1}^{n} \mathrm{E}(U_j) + Y_k \\
&= Y_k
\end{aligned}$$

Things to notice:

Y_k treated as constant given $Y_1, \ldots, Y_k$.

Knowing $Y_1, \ldots, Y_k$ is equivalent to knowing $U_1, \ldots, U_k$.

For $j > k$ we have U_j independent of $U_1, \ldots, U_k$ so conditional expectation is unconditional expectation.

Since Standard Brownian Motion is limit of such random walks we get martingale property for standard Brownian motion.

Poisson Process: $X(t) = N(t) - \lambda t$. Fix $t > s$.

$$\begin{aligned}
\mathrm{E}(X(t)|X(u); 0 \le u \le s) \\
&= \mathrm{E}(X(t) - X(s) + X(s)|\mathcal{H}_s) \\
&= \mathrm{E}(X(t) - X(s)|\mathcal{H}_s) + X(s) \\
&= \mathrm{E}(N(t) - N(s) - \lambda(t-s)|\mathcal{H}_s) + X(s) \\
&= \mathrm{E}(N(t) - N(s)) - \lambda(t-s) + X(s) \\
&= \lambda(t-s) - \lambda(t-s) + X(s) \\
&= X(s)
\end{aligned}$$

Things to notice:

I used independent increments.

$\mathcal{H}_s$ is shorthand for the conditioning event.

Similar to random walk calculation.

Black Scholes

We model the price of a stock as

$$X(t) = x_0 e^{Y(t)}$$

where

$$Y(t) = \sigma B(t) + \mu t$$

is a Brownian motion with drift (B is standard Brownian motion).

If annual interest rates are $e^{\alpha} - 1$ we call α the instantaneous interest rate; if we invest \$1 at time 0 then at time t we would have $e^{\alpha t}$. In this sense an amount of money $x(t)$ to be paid at time t is worth only $e^{-\alpha t}x(t)$ at time 0 (because that much money at time 0 will grow to $x(t)$ by time t).

Present Value: If the stock price at time t is $X(t)$ per share then the present value of 1 share to be delivered at time t is

$$Z(t) = e^{-\alpha t} X(t)$$

With X as above we see

$$Z(t) = x_0 e^{\sigma B(t) + (\mu - \alpha)t}$$

Now we compute

$$\begin{aligned} &\mathsf{E}\left\{Z(t) | Z(u); 0 \le u \le s\right\} \\ &\qquad = \mathsf{E}\left\{Z(t) | B(u); 0 \le u \le s\right\} \end{aligned}$$

for $s < t$. Write

$$Z(t) = x_0 e^{\sigma B(s) + (\mu - \alpha)t} \times e^{\sigma(B(t) - B(s))}$$

Since B has independent increments we find

$$\begin{aligned} &\mathsf{E}\left\{Z(t) | B(u); 0 \le u \le s\right\} \\ &\qquad = x_0 e^{\sigma B(s) + (\mu - \alpha)t} \times \mathsf{E}\left[e^{\sigma\{B(t) - B(s)\}}\right] \end{aligned}$$

Note: $B(t) - B(s)$ is $N(0, t-s)$; the expected value needed is the moment generating function of this variable at σ.

Suppose $U \sim N(0,1)$. The Moment Generating Function of U is

$$M_U(r) = \mathsf{E}(e^{rU}) = e^{r^2/2}$$

Rewrite

$$\sigma\{B(t) - B(s)\} = \sigma(t-s)U$$

where $U \sim N(0,1)$ to see

$$\mathsf{E}\left[e^{\sigma\{B(t)-B(s)\}}\right] = e^{\sigma^2(t-s)/2}$$

Finally we get

$$\begin{aligned}\mathsf{E}\{Z(t)|Z(u); 0 \le u \le s\} \\ &= x_0 e^{\sigma B(s)+(\mu-\alpha)s} e^{(\mu-\alpha)(t-s)+\sigma^2(t-s)/2} \\ &= Z(s)\end{aligned}$$

provided

$$\mu + \sigma^2/2 = \alpha\,.$$

If this identity is satisfied then the present value of the stock price is a martingale.

Option Pricing

Suppose you can pay \$$c$ today for the right to pay K for a share of this stock at time t (regardless of the actual price at time t).

If, at time t, $X(t) > K$ you will**exercise** your **option** and buy the share making $X(t) - K$ dollars.

If $X(t) \leq K$ you will not exercise your option; it becomes worthless.

The present value of this option is

$$e^{-\alpha t}(X(t) - K)_{+} - c$$

where

$$z_{+} = \begin{cases} z & z > 0 \\ 0 z \leq 0 & \end{cases}$$

(Called **positive part** of z.)

In a fair market:

- The discounted share price $e^{-\alpha t}X(t)$ is a martingale.

- The expected present value of the option is 0.

So:

$$c = e^{-\alpha t}\mathsf{E}\left[\{X(t) - K\}_+\right]$$

Since

$$X(t) = x_0 e^{N(\mu t, \sigma^2 t)}$$

we are to compute

$$\mathsf{E}\left\{\left(x_0 e^{\sigma t^{1/2} U + \mu t} - K\right)_+\right\}$$

This is

$$\int_a^\infty \left(x_0 e^{bu+d} - K\right) e^{-u^2/2} du/\sqrt{2\pi}$$

where

$$\begin{aligned} a &= (\log(K/x_0) - \mu t)/(\sigma t^{1/2}) \\ b &= \sigma t^{1/2} \\ d &= \mu t \end{aligned}$$

Evidently

$$K \int_a^\infty e^{-u^2/2} du/\sqrt{2\pi} = KP(N(0,1) > a)$$

The other integral needed is

$$\begin{aligned} \int_a^\infty e^{-u^2/2+bu} du/\sqrt{2\pi} \\ &= \int_a^\infty \frac{e^{-(u-b)^2/2} e^{b^2/2}}{\sqrt{2\pi}} du \\ &= \int_{a-b}^\infty \frac{e^{-v^2/2} e^{b^2/2}}{\sqrt{2\pi}} dv \\ &= e^{b^2/2} P(N(0,1) > a - b) \end{aligned}$$

Introduce the notation

$$\Phi(v) = P(N(0,1) \le v) = P(N(0,1) > -v)$$

and do all the algebra to get

$$\begin{aligned} c &= \left\{ e^{-\alpha t} e^{b^2/2+d} x_0 \Phi(b-a) - K e^{-\alpha t} \Phi(-a) \right\} \\ &= x_0 e^{(\mu+\sigma^2/2-\alpha)t} \Phi(b-a) - K e^{-\alpha t} \Phi(-a) \\ &= x_0 \Phi(b-a) - K e^{-\alpha t} \Phi(-a) \end{aligned}$$

This is the Black-Scholes option pricing formula.