Queuing Theory

Ingredients of Queuing Problem:

1: Queue input process.

2: Number of servers

3: Queue discipline: first come first serve? last in first out? pre-emptive priorities?

4: Service time distribution.

Example: Imagine customers arriving at a facility at times of a Poisson Process N with rate λ. This is the input process, denoted M (for Markov) in queuing literature.

Single server case:

Service distribution: exponential service times, rate μ.

Queue discipline: first come first serve.

$X(t)$ = number of customers in line at time t.

X is a Markov process called $M/M/1$ queue:

$$v_i = \lambda + \mu 1(i > 0)$$

$$\mathbf{P}_{ij} = \begin{cases} \frac{\mu}{\mu+\lambda} & j = i - 1 \geq 0 \\ \frac{\lambda}{\mu+\lambda} & j = i + 1, i > 0 \\ 1 & j = 1, i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Example: $M/M/\infty$ queue:

Customers arrive according to PP rate λ. Each customer begins service immediately. $X(t)$ is number being served at time t. X is a birth and death process with

$$v_n = \lambda + n\mu$$

and

$$\mathbf{P}_{ij} = \begin{cases} \frac{i\mu}{i\mu+\lambda} & j = i-1 \geq 0 \\ \frac{\lambda}{i\mu+\lambda} & j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Stationary distributions?

For $M/M/1$ queue:

Solve

$$\{\lambda + \mu 1(n > 0)\}\pi_n = \mu\pi_{n+1} + \lambda 1(n > 0)\pi_{n-1}$$

Just use general birth and death process formulation:

$$\lambda_n = \lambda \quad \mu_n = \mu 1(n > 0)$$

so

$$\frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = (\lambda/\mu)^n$$

and

$$\sum_{n=0}^{\infty} (\lambda/\mu)^n = 1/(1 - \lambda/\mu)$$

so

$$\pi_n = \frac{(\lambda/\mu)^n}{1 + 1/(1 - \lambda/\mu)}$$

Exists only if $\lambda < \mu$.

For $M/M/\infty$ queue:

$$\pi_n \propto \frac{\lambda^n}{\mu^n n!}$$

and

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{\mu^n n!} = \exp(\lambda/\mu)$$

so

$$\pi_n = \exp(-\lambda/\mu)\frac{\lambda^n}{\mu^n n!}$$

Notice this exists for all $\lambda > 0$ and all $\mu > 0$.

Scope of Queuing Theory:

1) $M/M/k$ queues. $X(t)$ is number queued or in service.

Birth and Death process; death rate maxes out at $k\mu$.

Stationary distribution exists if $\lambda < k\mu$.

2) Same input / service processes as $M/M/k$ but customers not served leave. Question of interest: customers lost per time unit?

Take X to be number in service. ($0 \le X(t) \le k$).

Find stationary distribution.

Fraction of time spent in state k is π_k.

During time in state k lose customers at rate λ. So lost $\pi_k\lambda$ customers per unit time.

3) $G/M/1$ queue. General distribution of interarrival times for input. Input is a **renewal process**. Not Markov.

4) $M/G/1$ and others.

5) Networks: output of 1 queue is input of next; feedback ...

Quantities of potential interest:

Average fraction of time server idle.

Average time in system for customer.

Average wait to see server.

One example calculation: in $G/M/1$ queue.

Compute long run fraction time system is idle.

Idea: interarrival times are iid with cdf G.

Service rate μ.

Let X_n be number of customers in service / in line when nth customer arrives.

Claim X_n is a Markov chain.

(Example of general tactic: find simple process buried within process of interest.)

Notation: $T_1, T_2, \cdots$ iid interarrival times.

Given $X_n = i$ and $T_{n+1} = t$ number served between nth arrival and $n+1$st arrival is

$$\min\{\text{Poisson}(\mu t), i+1\}$$

So: if $X_n = i$ and the Poisson variable above is j then

$$X_{n+1} = i + 1 - \min\{j, i+1\}$$

Now to compute prob of j served must average over T_{n+1}:

$$P(j \text{ served}) = \int e^{-\mu t}\frac{(\mu t)^j}{j!} dG(t) \equiv a_j$$

for $j \le i+1$.

This gives:

$$P_{ik} = \begin{cases} a_{i+1-k} & 1 \le k \le i+1 \\ 1 - \sum_0^i a_j & k = 0 \\ 0 & \text{otherwise} \end{cases}$$