Probability Definitions

Probability Space (or **Sample Space**): ordered triple $(\Omega, \mathcal{F}, P)$.

- Ω is a set (possible outcomes).

- $\mathcal{F}$ is a family of subsets (**events**) of Ω with the property that $\mathcal{F}$ is a σ-field (or Borel field or σ-algebra):

  1. The empty set $\emptyset$ and Ω are members of $\mathcal{F}$.

  2. $A \in \mathcal{F}$ implies $A^c = \{\omega \in \Omega : \omega \notin A\} \in \mathcal{F}$

  3. $A_1, A_2, \cdots$ all in $\mathcal{F}$ implies

$$A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}\,.$$

- P a function, domain $\mathcal{F}$, range a subset of $[0, 1]$ satisfying:

  1. $P(\emptyset) = 0$ and $P(\Omega) = 1$.

  2. **Countable additivity**: $A_1, A_2, \cdots$ **pairwise disjoint** $(j \neq k \implies A_j A_k = \emptyset)$

  $$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Axioms guarantee can compute probabilities by usual rules, including approximation without contradiction.

Consequences:

1. **Finite additivity** $A_1, \cdots, A_n$ pairwise disjoint:

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)\,.$$

2. For any event A $P(A^c) = 1 - P(A)$.

3. If $A_1 \subset A_2 \subset \cdots$ are events then

$$P(\bigcup_{1}^{\infty} A_i) = \lim_{n\to\infty} P(A_n)\,.$$

4. If $A_1 \supset A_2 \supset \cdots$ then

$$P(\bigcap_{1}^{\infty} A_i) = \lim_{n\to\infty} P(A_n)\,.$$

Most subtle point is σ-field, $\mathcal{F}$. Needed to avoid some contradictions which arise if you try to define $P(A)$ for every subset A of Ω when Ω is a set with uncountably many elements.

Events in Set Notation

Event that Y_n converges to 0 is

$$A \equiv \{\omega : \lim_{n\to\infty} Y_n(\omega) = 0\}$$

Not explicitly written in terms of simple events involving only a finite number of Ys.

Recall basic definition of limit: y_n converges to 0 if $\forall \epsilon > 0$ $\exists N$ such that $\forall n \geq N$ we have $|y_n| \leq \epsilon$.

Convert the definition in A into set theory notation:

- replace y_n by $Y_n(\omega)$,

- replace each *for every* by an intersection

- replace each *there exists* with a union.

We get

$$A = \bigcap_{\epsilon>0} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega : |Y_n(\omega)| \le \epsilon\}$$

Not obvious A is event because intersection over $\epsilon > 0$ is uncountable.

However, the intersection is countable. Let

$$B_\epsilon \equiv \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega : |Y_n(\omega)| \leq \epsilon\}$$

Notice that

$$\epsilon' < \epsilon \implies B_{\epsilon'} \subset B_\epsilon$$

This means that

$$\bigcap_{\epsilon>0} B_\epsilon = \bigcap_{m=1}^{\infty} B_{1/m}$$

A is countable intersection of countable unions of countable intersections of events, so A is an event.

Here are some other events:

Sequence S_n has a limit. Sequence s_n has a limit if $\exists s_\infty$ such that $\forall \epsilon > 0$ $\exists N$ such that $\forall n \geq N$ we have $|s_n - s_\infty| \leq \epsilon$. Mechanically get event:

$$\bigcup_s \bigcap_{\epsilon>0} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega : |S_n(\omega) - s| \leq \epsilon\}$$

Intersection over ϵ can be made countable. Union over s, however, is not easy to make countable. Instead use theorem of analysis to describe existence of a limit.

A sequence s_n has a limit if and only if the sequence is Cauchy.

Cauchy sequence: $\forall \epsilon > 0$ $\exists N$ such that $\forall n \geq N$ we have $|s_n - s_N| \leq \epsilon$. $\{S_n$ has a limit$\}$ is

$$\bigcap_{\epsilon>0} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|S_n - S_N| \leq \epsilon\}$$

Intersection over all $\epsilon > 0$ is countable intersection over $\epsilon = 1/r$ for positive integers r.

Y_n **is summable**: the sequence of partial sums $S_n = \sum_1^n Y_i$ has a limit so

$$\{Y_n \text{ summable}\} = \bigcap_{r=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\left|\sum_{N+1}^{n} Y_j\right| \leq 1/r\}$$

Event $S_n > 0$ **for infinitely many** n: $\forall N \exists n \geq N$ such that $S_n > 0$. is

$$\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{S_n > 0\}$$

limit superior of S_n is 1 is intersection of two events, $\limsup S_n \leq 1$ and $\limsup S_n \geq 1$. Former is $\forall \epsilon > 0 \exists N$ such that $\forall n \geq N$ $S_n \leq 1 + \epsilon$. Latter is $\forall \epsilon > 0$ and $\forall N$ there is an $n \geq N$ such that $S_n \geq 1 - \epsilon$. Event is $A^* \cap A_*$ where

$$A^* = \bigcap_{r=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{S_n \leq 1 + 1/r\}$$

$$A_* = \bigcap_{r=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{S_n \geq 1 + 1/r\}$$

Random Variables:

Vector valued random variable: function X, domain Ω, range in $\mathbb{R}^p$ such that

$$P(X_1 \le x_1, \ldots, X_p \le x_p)$$

is defined for any constants $(x_1, \ldots, x_p)$. Notation: $X = (X_1, \ldots, X_p)$ and

$$X_1 \le x_1, \ldots, X_p \le x_p$$

is shorthand for an event:

$$\{\omega \in \Omega : X_1(\omega) \le x_1, \ldots, X_p(\omega) \le x_p\}$$

X function on Ω so X_1 function on Ω.

Formal definitions:

The **Borel** σ-field in $\mathbb{R}^p$ is the smallest σ-field in $\mathbb{R}^p$ containing every open ball

$$B_y(r) = \{x \in \mathbb{R}^p : |x - y| < r\} \, .$$

(To see that there is, in fact, such a "smallest" σ-field you prove the following assertions:

1. The intersection of an arbitrary family of σ-fields is a σ-field.

2. There is at least one σ-field of subsets of $\mathbb{R}^p$ containing every open ball.

Now define the Borel σ-field in $\mathbb{R}^p$ to be

$$\mathcal{B}(\mathbb{R}^p) = \cap \mathcal{F}$$

where the intersection runs over all σ-fields $\mathcal{F}$ which contain every open ball.)

Every common set is a Borel set, that is, in the Borel σ-field.

Example: If O is an open set then O is Borel.

Proof: For each x in O there is a point y all of whose co-ordinates are rational numbers and a rational number r such that

$$x \in B_y(r) \subset O$$

Now O is the union of all these $B_y(r)$.

(Every $x \in O$ is in one of the $B_y(r)$ and every point in any $B_y(r)$ is in O.)

But the union is countable because there are only countably many possible pairs (y, r) with all the co-ordinates rational numbers.