Product Spaces

Suppose $\mathcal{X}_i, \mathcal{F}_i$ are pairs for $i = 1, \ldots, p$.

Each $\mathcal{X}_i$ a set; $\mathcal{F}_i$ a σ-field of subsets of $\mathcal{X}_i$.

The Cartesian product of the sets $\mathcal{X}_i$ is

$$\mathcal{X} = \{(x_1, \ldots, x_p) : x_i \in \mathcal{X}_i; i = 1, \ldots, p\}$$

We write

$$\mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_p$$

We define a subset A of $\mathcal{X}$ to be a measurable rectangle if

$$A = A_1 \otimes \cdots \otimes A_p$$

where each A_i is in $\mathcal{F}_i$.

We define the product σ-field on $\mathcal{X}$ as

$$\mathcal{F} = \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_p$$

where $\mathcal{F}$ is the smallest σ-field containing all measurable rectangles.

Note: *smallest* means intersection of all σ-fields containing the family of measurable rectangles. This intersection is not empty (consider power set of $\mathcal{X}$). And any intersection of σ-fields is a σ-field.

How do we prove independence result:

Step 1: True for pairs A, B where both A and B are rectangles by definition of independence.

Step 2: Fix rectangle B. Consider collection of sets A for which (**??**) holds. It is closed under finite disjoint unions. It is closed under the action of taking complements. It contains the all measurable rectangles. So it contains the smallest field containing all measurable rectangles.

(Field is like σ field but with finite unions and intersections.)

The collection of sets A for which (**??**) holds is a monotone class. So it contains the smallest monotone class containing the field generated by the rectangles. So it contains the σ-field generated by the rectangles – the product σ-field.

Monotone Class: collection $\mathcal{C}$ of subsets of a given set, closed under increasing countable unions and decreasing countable intersections:

1. If $A_1 \subset A_2 \subset \cdots$ are in $\mathcal{C}$ then $\bigcup_1^\infty A_i \in \mathcal{C}$.

2. If $A_1 \supset A_2 \supset \cdots$ are in $\mathcal{C}$ then $\bigcap_1^\infty A_i \in \mathcal{C}$.

Lemma: The smallest monotone class containing a field $\mathcal{F}_o$ is a σ-field.

Proof of Lemma: Let $\mathcal{C}$ be smallest monotone class containing $\mathcal{F}_o$ (intersection of all monotone classes containing $\mathcal{F}_o$). Put $\mathcal{M} = \{A \in \mathcal{C} : A^c \in \mathcal{C}\}$.

- $\mathcal{M}$ is a monotone class.

- Since $\mathcal{F}_o$ is a field and a subset of $\mathcal{C}$, $\mathcal{M}$ contains $\mathcal{F}_o$.

- Since $\mathcal{C}$ is smallest monotone class containing $\mathcal{F}_o$ and $\mathcal{M}$ is another monotone class containing $\mathcal{F}_o$ we see $\mathcal{C} \subset \mathcal{M}$.

- This means every $A \in \mathcal{C}$ has the property $A^c \in \mathcal{C}$. In other words $\mathcal{C}$ is closed under the operation of taking complements, one of the defining properties of a σ-field.

- Similar argument for finite unions in notes.

- So $\mathcal{C}$ is a field. Since $\mathcal{C}$ is monotone class and field $\mathcal{C}$ is a σ-field.

Related mathematical problem. We use independence two ways: as a modelling tactic and as a computational tool.

We often model by assuming some random variables are independent.

Suppose $X_1, X_2, \cdots$ are an infinite sequence of independent coin tosees?

But can we suppose so? Does there exist $(\Omega, \mathcal{F}, \mathbf{P})$ and random variables $X_1, X_2, \cdots$ defined on Ω such that they are independent and 0, 1 valued?

Yes: use extension theorems.

Product measures:

Suppose P_i is a probability measure on $(\Omega_i, \mathcal{F}_i)$. We define a product measure on

$$\Omega = \Omega_1 \otimes \cdots \otimes \Omega_P$$

by

$$P(A_1 \otimes \cdots \otimes A_p) = \prod P_i(A_i)$$

This formula extends to the product σ-field by say the Caratheodory extension theorem.

Use: the maps $X_i : \Omega \mapsto \Omega_i$ given by

$$X_i(\omega_1, \ldots, \omega_p) = \omega_i$$

These rvs are independent and $P(X_i \in A_i) = P_i(A_i)$.

Can even take $p = \infty$ using Kolmogorov consistency theorem.