Diffusions

Next natural generalization: continuous time, continuous state space Markov Chains: $S = \mathbb{R}$.

Study processes $\{X(t); t \geq 0\}$ where each $X(t)$ is real valued and $t \mapsto X(t)$ is continuous.

Idea for diffusions: model short time behaviour: given $\mathcal{H}_t$

$$X(t+h) = X(t) + \mu(X(t))h + \sigma(X(t))\sqrt{h}\epsilon + o(h)$$

where ϵ is a standard normal variable and $\mu(\cdot)$ and $\sigma(\cdot)$ are model specified functions.

Meaning: the equation is an assertion about the approximate distribution of $X(t+h) - X(t)$.

Chapman-Kolmogorov. Let $f(t, x, y)$ be the density of $X(t)$ at y given $X(t) = x$. Then

$$f(t+s, x, y) = \int_{-\infty}^{\infty} f(s, x, z) f(t, z, y) dz$$

Kolmogorov's Backward Equations

Informal presentation.

Let h be small. Assume that

$$f(h,x,y) \approx \frac{\text{exp}\{-(y-x-\mu(x)h)^2/(2\sigma^2(x)h)\}}{\sqrt{2\pi h}\sigma(x)}$$

That is, we assume that $X(h)$ is, given $X(0)$, approximately normally distributed with mean $X(0)+\mu(X(0))h$ and variance $h\sigma^2(X(0))$. Then

$$f(t+h,x,y) = \int f(h,x,z)f(t,z,y)dz\,.$$

Make the change of variables

$$u = \frac{z - x - \mu(x)h}{\sigma(x)\sqrt{h}}$$
$$du = \frac{dz}{\sigma(x)\sqrt{h}}$$

to find

$$f(t+h,x,y) = \int \frac{e^{-u^2/2}}{\sqrt{2\pi}} f(t, x + \mu(x)h + \sigma(x)u\sqrt{h}, y)du$$

Suppose f is twice differentiable wrt x: Taylor expansion:

$$\begin{aligned} f(t,&x + \mu(x)h + \sigma(x)u\sqrt{h}, y) \\ &= f(t,x,y) + \frac{\partial f}{\partial x}(\mu(x)h + \sigma(x)u\sqrt{h}) \\ &\qquad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu(x)h + \sigma(x)u\sqrt{h})^2 + o(h) \end{aligned}$$

Partial derivatives evaluated at (t, x, y).

Put these into Chapman Kolmogorov and use

$$\int u \frac{e^{-u^2/2}}{\sqrt{2\pi}} du = 0$$

to get

$$f(t+h,x,y) = f(t,x,y) + \mu(x)h\frac{\partial f}{\partial x} + \frac{\sigma^2(x)}{2}h\frac{\partial^2 f}{\partial x^2}$$

Move $f(t,x,y)$ to other side, divide by h and take limit:

$$\frac{\partial f}{\partial t} = \mu(x)\frac{\partial f}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2 f}{\partial x^2}$$

This differential equation is called a diffusion equation.

Special case: $\mu \equiv 0$ and $\sigma \equiv 1$ gives the heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$$

Solving this equation?

Consider simpler case: Compute

$$H(t,x) \equiv \mathsf{E}^x\left[\phi(X_t)\right]$$

for some nice function ϕ.

Multiply Chapman-Kolmogorov equations by $\phi(y)$ and integrate dy.

Discover H also solves heat equation:

$$\frac{\partial H}{\partial t} = \frac{1}{2}\frac{\partial^2 H}{\partial x^2}$$

with initial condition:

$$H(0,x) = \phi(x)$$

Solved by taking Fourier transforms. Get:

$$H(t,x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/(2t)}\phi(y)dy$$

Interpretation: X_t is $N(x,t)$.

Another interpretation: $f(t,x,y)$ is normal density centered at X with standard deviation $\sqrt{t}$.