Modelling Traffic Loading on the Lion's Gate Bridge

Idea: want to know how strong bridge needs to be.

Compute: load x such that

Expected time to first exceedance of load x is 100 years.

Method uses:

1) modelling assumptions.

2) conservative modelling; to replace random variable of interest with stochastically larger quantity.

3) moment generating functions; Markov's inequality — compute upper bound on x.

Facts about the bridge:

Built 1936-38 for $6M.

3 spans: 614, 1550, 614 feet long.

Originally 2 lanes now 3.

Originally toll bridge built by developers.

See

`http://www.b-t.com/projects/liongate.htm`

at Buckland and Taylor web site for engineering info.

Begin with definition of process of interest.

Think of bridge as rectangle.

Co-ordinates: x runs from 0 to length L_B of bridge, y runs from 0 to width W_B of bridge.

Define:

$Z(x, y, t) =$ load on bridge at (x, y) at time t

General quantity of interest at time t: total load or other force on segment of bridge:

$$\int_{xy} Z(x, y, t) w(x, y) dx dy$$

Example w: load in strip across bridge between x_1 and x_2 feet out from south side on central span

$$W(t, x_1, x_2) = \int_{x_1}^{x_2} \int_0^{W_B} Z(x, y, t) dy dx$$

Quantity of concern to engineers:

$$M_T(L) \equiv \max_{t \in [0,T]} \max_{0 \le x_1 \le L_B - L} W(t, x_1, x_1 + L)$$

First modelling assumption. Years $1, \ldots, T$ are iid.

So:

$$P(M_T(L) \le y) = P(M_1(L) \le y)^T$$

So: years to first exceedance of level y has geometric distribution with probability of success

$$P(M_1(L) > y)$$

Find y so this last is 1/100; expected value of geometric is 100.

Call this the 100 year return time load.

Next modelling consideration.

Two kinds of loads: static and dynamic.

Consider only static not dynamic loading.

Observation: static loading much higher when traffic stopped than not.

So: define N to be number of traffic stoppages in year.

Let $M_{1,n}(L)$ be worst load over segment of length L during nth of N stoppages.

Idea

$$P(M_1(L) > y) = P(\max_{1 \le n \le N} M_{1,n}(L) > y)$$

Treat $M_{1,n}(L)$ as iid *given* N.

Next: Evaluate $P(M_1(L) > y)$ by conditioning.

Shorten notation:

$$M = \max_{1 \le n \le N} X_n$$

where X_i iid, cdf F, survival ftn $S = 1 - F$.

$$\begin{aligned} P(M \le y) &= \mathsf{E}\{P(M \le y|N)\} \\ &= \mathsf{E}[\{1 - S(y)\}^N)] \\ &= \phi[\log\{1 - S(y)\}] \end{aligned}$$

where

$$\phi(t) = \mathsf{E}\left(e^{tN}\right)$$

is the moment generating function of N.

Comment: ϕ is monotone increasing.

So: if $S(y) \leq g(y)$ then

$$P(M > y) \leq 1 - \phi[\log\{1 - g(y)\}]$$

and solving

$$\phi[\log\{1 - g(y)\}] = 0.99$$

gives larger solution than

$$P(M \leq y) = 0.99$$

Remaining steps:

1) Model for N.

2) Model / upper bound for S.

Modelling N:

Simplest idea: Poisson process of accidents.

So N has Poisson(λ) dist for some λ.

Then

$$\phi(t) = \sum e^{-\lambda} \frac{\left(\lambda e^t\right)^n}{n!}$$

which is

$$\phi(t) = \exp\{\lambda(e^t - 1)\}$$

Criticisms:

No allowance for variation in traffic densities, weather, etc from year to year.

Potentially better assumption.

N is overdispersed Poisson, say, Negative Binomial:

$$P(N = k) = \binom{r+k-1}{k} p^r (1-p)^k \quad k = 0, 1, \cdots$$

This makes

$$\mathrm{E}(e^{tX}) = \frac{p^r}{\{1-(1-p)e^t\}^r}$$

Idea: for Poisson $\sigma = \sqrt{\mu}$.

For Negative Binomial $\mu = r(1-p)/p$ and

$$\sigma = \sqrt{r(1-p)/p^2} > \sqrt{1/p}\sqrt{\mu} > \sqrt{\mu}$$

Idea: use of overdispersed variable makes for longer tails relative to mean.

Now we need to model / bound the survival function S.

Stoppage lasts random time T.

During that time traffic builds up behind stoppage; cars jam together.

Worst section of length L found by sliding window along line of stopped cars to find maximum.

Notional model (not the way we did it):

Model vehicles arriving at end of queue.

Might use Poisson Process.

Each vehicle has random mass, length, distribution of load along length.

Random gaps between vehicle.

Just before traffic starts to move again:

Look for heaviest segement of length L in stoppage.

Problems:

1) different kinds of stoppage: # lanes, direction of flow, location on bridge, cars trickle past?

2) hard to deal with supremum over all segments of length L.

3) specify joint law of mass, length, distribution of load along single vehicle.

Digression to method we didn't use:

Model length of stoppage T with density g.

Model N, number of vehicles arriving at end of stoppage, given T as Poisson(λT).

Assume next vehicle arriving picked at random; joint density $h(w,l)$ of weight, length. W_i, L_i values for ith arrival.

Assume load distributed evenly along length of vehicle.

Final length of line at end of stoppage is

$$L_T \equiv \sum_{i=1}^{N} L_i.$$

Can compute mean, variance, generating function of L_T?

$$\begin{aligned} \mathrm{E}\{\exp(sL)\} &= \mathrm{E}\left[\mathrm{E}\{\exp(sL)|N\}\right] \\ &= \mathrm{E}\left([\mathrm{E}\{\exp(sL_i)\}]^N\right) \\ &= \phi_N[\log\{\phi_L(s)\}] \end{aligned}$$

Here each ϕ is a moment generating function.

This is a method of analysis for a compound Poisson Process. Can use the mgf of L to compute distribution of L by inversion of Laplace transform.

Problem: how to scan for maximum load?

Simplify problem: discretize and bound.

General idea discretization.

If h is small and $X(s)$ some process then

$$\sup_{0 \le t \le T-\tau} \int_t^{t+\tau} X(s) ds$$

is close to

$$\max_k \int_{kh}^{kh+\tau} X(s) ds$$

We took h to be 50 feet.

Considered $\tau = 50n$ feet.

Switch from thinking about length of stoppage in time to length of stoppage in multiples of h.

Let N_i be number of segments of length h building up on bridge during stoppage

Let $X_j; j = 1, \ldots, N_i$ be the loads on the consecturive segments.

So: our interest is in

$$\max\{X_r + \cdots X_{r+n-1}; 1 \le r \le N_i + 1 - n\}$$

Upper bound on survival function?

Define

$$U_r = X_r + \cdots X_{r+n-1}$$

Argue that

$$1 - P(\max_{1 \le r \le m} U_r > y) \le 1 - \prod_{1 \le r \le m} \{1 - P(U_r > y)\}$$

for any m.

Rationale: Values of U_r are *positive orthant dependent*.

(Large values of one U suggest large values of adjacent U.)

So now:

$$P(\max\{U_r; 1 \le r \le N_i\} > y) \le 1 - \phi_{N_i}[\log\{1 - S_U(y)\}]$$

Final step we took.

Model law of X_i by considering possible loading patterns by cars, trucks, buses.

We took cars to be fixed length and weight.

Same for buses.

Trucks had fixed length, weight uniform on 12 to 40 tons.

Computed moment generating function of an X:

$$\phi_X(t) = \mathsf{E}(e^{tX})$$

Final step. Need to compute S_U.

Instead use Markov's inequality:

$$P(X \geq x) \leq \frac{\mathrm{E}\{g(X)\}}{g(x)}$$

for any increasing positive g.

Choose $g(\cdot) = \exp(h\cdot)$.

So:

$$\begin{aligned} S_U(y) &\leq \frac{\mathrm{E}(e^{hU})}{\exp(hy)} \\ &= \frac{\{\phi_X(h)\}^n}{\exp(hy)} \end{aligned}$$

where

$$\phi_X(h) = \mathrm{E}\{\exp(hX)\}$$

These can be assembled to give a bound depending on h.

Then: minimize over $h > 0$ to find good bound.