Renewal Theory

Basic idea: study processes where after random time everything starts over at the beginning.

Example: M/G/1 queue starts over every time the queue empties.

Begin with **renewal process**:

Have counting process $N(t)$.

Times between arrivals are $T_1, T_2, \cdots$

Time of nth arrival is

$$S_n = \sum_{i=1}^{n} T_i$$

If arrival times iid with distribution F call N a renewal process.

Poisson process is example with F an exponential cdf.

Define $N(t) =$ number of renewals by time t.

So $N(t) = k$ if and only if

$$S_k \leq t < S_{k+1}$$

So:

$$\begin{aligned} P(N(t) = k) &= P(S_k \leq t < S_{k+1}) \\ &= P(S_k \leq t) - P(S_k \leq t \cap S_{k+1} \leq t) \\ &= P(S_k \leq t) - P(S_{k+1} \leq t) \end{aligned}$$

Jargon: cdf of sum of k iid T_i is called **convolution**.

Basic principles:

In the long run the process forgets its starting time.

Long run renewal rate is $1/\mu$ where μ is the expected lifetime of one X.

Instantaneous renewal rate is eventually $1/\mu$. (Not conditional!)

Mean values: define $m(t) = \mathsf{E}(N(t))$.

$$\begin{aligned} m(t) &= \mathsf{E}(N(t)) \\ &= \sum_k kP(N(t) = k) \\ &= \sum_k P(N(t) \ge k) \\ &= \sum_k P(S_k \le t) \end{aligned}$$

Fact: m is finite.

Proof: Find c so that $p = P(T_1 \le c) < 1$.

Success: $T_i \le c$.

Failure: $T_i > c$.

$B = \#$ Successes $\sim$ Binomial(n, p).

If $n - B > t/c$ then $S_n > t$.

So

$$\begin{aligned} P(S_n \le t) &\le P(B > n - t/c) \\ &= P(e^{\lambda B} \le e^{\lambda(n-t/c)}) \\ &\le \frac{E(e^{\lambda B})}{e^{\lambda(n-t/c)}} \\ &= e^{t/c}(pe^{\lambda} + 1 - p)^n e^{-\lambda n} \\ &= e^{t/c}\left\{p + (1-p)e^{-\lambda}\right\}^n \end{aligned}$$

This is summable.

In fact compute m by conditioning on T_1:

$$\mathsf{E}(N(t)) = \mathsf{E}\left[\mathsf{E}(N(t)|T_1)\right]$$

If $x > t$ and we are given $T_1 = x$ then $N(t) = 0$.

If $x \le t$ and we are given $T_1 = x$ then $N(t)$ has the same law as

$$1 + N(t - x)$$

so for $x \le t$

$$\mathsf{E}[N(t)|T_1 = x] = 1 + m(t - x)$$

This makes

$$\mathsf{E}(N(t)|T_1) = \{1 + m(t - T_1)\}\, 1(T_1 \le t)$$

Take expected values: **Renewal equation**

$$m(t) = F(t) + \mathsf{E}[m(t - T_1)1(T_1 \le t)]$$

If F has density f

$$m(t) = F(t) + \int_0^t m(t - x)f(x)dx.$$

Basic renewal limit theorems:

Let $\mu = \mathrm{E}(T_1)$.

First as $t \to \infty$:

$$N(t)/t \to 1/\mu$$

Second: the elementary renewal theorem:

$$m(t)/t \to 1/\mu$$

Note: not as easy to prove as it looks.

Example: if $f(x) = 1(0 < x < 1)$ then renewal equation says, for $0 < t < 1$:

$$m(t) = t + \int_0^t m(t-x)dx = t + \int_0^t m(x)dx$$

Differentiate:

$$m'(t) = 1 + m(t)$$

or

$$\log(1 + m(t)) = t + c$$

Put $t = 0$ to find $c = 0$ and

$$m(t) = e^t - 1 \qquad \text{for } 0 < t < 1$$

Not linear!

For $1 < t < 2$:

$$\begin{aligned} m(t) &= 1 + \int_0^1 m(t-x)dx \\ &= 1 + \int_{t-1}^t m(u)du \end{aligned}$$

Differentiate and solve to get

$$m(t) = e^{t-1}(1-t) + e^t - 1.$$

Regeneration:

Now consider a stochastic process with the property:

There is a random time T such that:

$$P(T < \infty) = 1$$

and such that at time T the process starts over: the conditional distribution of the future given T and everything happening up to time T is the unconditional distribution of the process started at time 0.

Called a **regeneration** (or **renewal**) time.

Gives rise to sequence of times $T_1, T_2, \cdots$ which are iid.

Let $N(t)$ denote number of renewals by time t.

Associate to each cycle some random variable R_k, iid. (Typically same function applied to path of process over one cycle.)

Define

$$R(t) = \sum_{i=1}^{N(t)} R_i$$

Basic facts:

$$\frac{R(t)}{t} = \frac{R(t)}{N(t)} \frac{N(t)}{t} \to \frac{\mathsf{E}(R_1)}{\mu}$$

and

$$\frac{\mathsf{E}[R(t)]}{t} = \frac{\mathsf{E}[R(t)]}{m(t)} \frac{m(t)}{t} \to \frac{\mathsf{E}(R_1)}{\mu}$$

Processes with regeneration times:

1) Recurrent Markov chains

2) M/G/1 queue with input rate less than output rate.

3) G/M/1 queue with input rate less than output rate.

Look at # 3: In each cycle think of B_i as busy time and I_i as idle time.

Total length of cycle is $B_i + I_i$.

Let $R(t)$ be amount of idle time up to time t.

Get:

$$\frac{R(t)}{t} = \!\!\rightarrow \frac{\mathsf{E}(I_1)}{\mathsf{E}(I_1 + B_1)}$$

and

$$\frac{\mathsf{E}[R(t)]}{t} = \!\!\rightarrow \frac{\mathsf{E}(I_1)}{\mathsf{E}(I_1 + B_1)}$$

Can we compute the pieces?

1) Number served from start of busy period to start of next busy period is N.

2) $T_1, T_2, \cdots$ interarrival times for input.

Total length of cycle is

$$\sum_{i=1}^{N} T_i$$

Fact: N is a stopping time ($\{N = n\}$ is independent of $T_{n+1}, \cdots$.

Wald's identity (added to homework):

$$\mathsf{E}\left[\sum_{i=1}^{N} T_i\right] = \mathsf{E}[N]\,\mathsf{E}[T_1]$$

Note $\mathsf{E}[T_1] = \int t dG(t)$.

(Notation in book: $\lambda^{-1} \int t dG(t)$.)

Compute $\mathsf{E}[N]$?

N is number of transitions of Markov chain between visits to state 0.

So $\pi_0 = 1/\mathsf{E}[N]$.

That is

$$\mathsf{E}[N] = 1/(1-\beta)$$

So expected cycle length is

$$\frac{1}{\lambda(1-\beta)}$$

Book presents following argument.

Let P_k denote fraction of time system has k people in line.

In steady state: transition rate from k to $k+1$ must balance reverse transition rate.

Downward rate is $P_{k+1}\mu$. (Proportion of time in state $k+1$ times service rate.)

Upward rate is average arrival rate times proportion of arrivals finding k in system or

$$\pi_k \lambda$$

Get, for $k \geq 0$

$$P_{k+1}\mu = \pi_k \lambda$$

Or

$$P_{k+1} = \frac{\lambda}{\mu}(1-\beta)\beta^k$$

Since $\sum_0^\infty P_k = 1$ can solve for P_0.

Formulas for solution in text.