STAT 870

Problems: Assignment 4

1. Show that if B_t is standard Brownian motion then $B_t^2 - t$ is a martingale.

2. Suppose that $B(t)$ is standard Brownian motion. Define

$$W(t) = \begin{cases} tB(1/t) & t > 0 \\ 0 & t = 0 \end{cases}$$

   Show that W is Brownian motion.

3. Suppose that $B(t)$ is a standard Brownian motion. Let $W(t) = B(t) - tB(1)$. Compute $\mathrm{E}(W(t))$ and $\mathrm{Cov}(W(s), W(t))$ for $0 < s < t < 1$.

4. Suppose that $B(t)$ is a standard Brownian motion. Let $W(t) = B(t) - tB(1)$. Compute the $\mathrm{E}(B(t)|B(1) = 0)$ and $\mathrm{Cov}(B(s), B(t)|B(1) = 0)$ for $0 < s < t < 1$. Compare this calculation with the result of the previous calculation. The process in this problem and the previous is called a **Brownian Bridge**.

5. Suppose $W(t)$ is the Brownian Bridge of problem 3/4. Let

$$Z(t) = (t + 1)W(t/(t + 1))$$

   and show that Z is Brownian Motion.

6. Use the definition of stochastic integral which I gave in class to argue that if f is a differentiable function on $[0, 1]$ and B is Brownian motion then

$$\int_0^1 f(t)dB(t) = f(1)B(1) - \int_0^1 B(s)f'(s)ds$$

   where the integral on the right is a Riemann integral – the usual kind.

7. Suppose that $M_0, M_1, \ldots, M_n$ is a (discrete time Martingale) meaning

$$E(M_{k+1}|M_0, \ldots, M_k) = M_k$$

for all $0 \le k \le n-1$. Suppose $T \in \{0, 1, \ldots, n\}$ is a stopping time, that is, that $1(T = k) = f_k(M_0, \ldots, M_k)$ for some functions f_k. Show that

$$E(M_T | M_0) = M_0.$$

(We say M_0, M_T is a martingale (with 2 terms). This is called **optional stopping** or **optional sampling**.)

8. Assume that the result in the previous problem holds for a continuous time martingale. Let B_t be Brownian motion and T be the first time that the process hits either $a > 0$ or $b < 0$. Compute

$$P(\text{Brownian motion hits } a \text{ before it hits } b).$$

Due: Monday 25 July, 2011.