

Diffusions

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Purposes of Today's Lecture

- Describe diffusion modelling assumptions.
- Derive Kolmogorov's backwards equation.
- Deduce partial differential equation.
- Discuss special case of heat equation.



Diffusions

- Next natural generalization: continuous time, continuous state space
Markov Chains: $S = \mathbb{R}$.
- Study processes $\{X(t); t \geq 0\}$ where each $X(t)$ is real valued and $t \mapsto X(t)$ is continuous.
- Idea for diffusions: model short time behaviour: given \mathcal{H}_t

$$X(t+h) = X(t) + \mu(X(t))h + \sigma(X(t))\sqrt{h}\epsilon + o(h)$$

where ϵ is a standard normal variable and $\mu(\cdot)$ and $\sigma(\cdot)$ are model specified functions.

- Meaning: the equation is an assertion about the approximate distribution of $X(t+h) - X(t)$.
- Chapman-Kolmogorov. Let $f(t, x, y)$ be the density of $X(t)$ at y given $X(0) = x$. Then

$$f(t+s, x, y) = \int_{-\infty}^{\infty} f(s, x, z)f(t, z, y)dz$$



Kolmogorov's Backward Equations

- Informal presentation.
- Let h be small. Assume that

$$f(h, x, y) \approx \frac{\exp\{-(y - x - \mu(x)h)^2 / (2\sigma^2(x)h)\}}{\sqrt{2\pi h\sigma(x)}}$$

- That is, we assume that $X(h)$ is, given $X(0)$, approximately normally distributed with mean $X(0) + \mu(X(0))h$ and variance $h\sigma^2(X(0))$.
- Then

$$f(t + h, x, y) = \int f(h, x, z)f(t, z, y)dz .$$



Backward Equations Continued

- Make the change of variables

$$u = \frac{z - x - \mu(x)h}{\sigma(x)\sqrt{h}} \text{ and } du = \frac{dz}{\sigma(x)\sqrt{h}}$$

to find

$$f(t+h, x, y) = \int \frac{e^{-u^2/2}}{\sqrt{2\pi}} f(t, x + \mu(x)h + \sigma(x)u\sqrt{h}, y) du$$

- Suppose f is twice differentiable wrt x : Taylor expansion:

$$\begin{aligned} & f(t, x + \mu(x)h + \sigma(x)u\sqrt{h}, y) \\ &= f(t, x, y) + \frac{\partial f}{\partial x}(\mu(x)h + \sigma(x)u\sqrt{h}) \\ & \quad + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(\mu(x)h + \sigma(x)u\sqrt{h})^2 + o(h) \end{aligned}$$

Partial derivatives evaluated at (t, x, y) .



Backward Equations Continued

- Put these into Chapman Kolmogorov and use

$$\int u \frac{e^{-u^2/2}}{\sqrt{2\pi}} du = 0$$

to get

$$f(t+h, x, y) = f(t, x, y) + \mu(x)h \frac{\partial f}{\partial x} + \frac{\sigma^2(x)}{2} h \frac{\partial^2 f}{\partial x^2} + o(h).$$

- Move $f(t, x, y)$ to other side, divide by h and take limit:

$$\frac{\partial f}{\partial t} = \mu(x) \frac{\partial f}{\partial x} + \frac{\sigma^2(x)}{2} \frac{\partial^2 f}{\partial x^2}.$$

- This differential equation is called a diffusion equation.
- Special case: $\mu \equiv 0$ and $\sigma \equiv 1$ gives the heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}$$



Solving the heat equation

- Consider simpler case: Compute

$$H(t, x) \equiv \mathbb{E}^x [\phi(X_t)]$$

for some nice function ϕ .

- Multiply Chapman-Kolmogorov equations by $\phi(y)$ and integrate dy .
- Discover H also solves heat equation:

$$\frac{\partial H}{\partial t} = \frac{1}{2} \frac{\partial^2 H}{\partial x^2}$$

with initial condition:

$$H(0, x) = \phi(x)$$

- Solved by taking Fourier transforms. Get:

$$H(t, x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/(2t)} \phi(y) dy$$

- Interpretation: X_t is $N(x, t)$.
- Another interpretation: $f(t, x, y)$ is normal density centered at X with standard deviation \sqrt{t} .

