The Lions Gate Bridge

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Purposes of Today's Lecture

- Illustrate modelling process.
- Show some probabilistic notation.
- Show potential use of inequalities.
- Introduce independence, overdispersion, Markov's inequality.

Traffic Loading on Lions Gate Bridge

- Idea: want to know how strong bridge needs to be.
- Compute: load x such that
  Expected time to first exceedance of load x is 100 years.
- Method uses:
  1. modelling assumptions.
  2. conservative modelling; to replace random variable of interest with stochastically larger quantity.
  3. moment generating functions; Markov's inequality — compute upper bound on x.

The Lions Gate Bridge

Facts about the Lions Gate Bridge

- Built 1936-38 for $6M.
- 3 spans: 614, 1550, 614 feet long.
- Originally 2 lanes now 3.
- Originally toll bridge built by developers.
- See

  http://www.b-t.com/projects/liongate.htm

  at Buckland and Taylor web site for engineering info.

The Lions Gate Bridge

Process of Interest

- Think of bridge as rectangle.
- Co-ordinates: x runs from 0 to length L_B of bridge, y runs from 0 to width W_B of bridge.
- Define:

$$Z(x, y, t) = \text{ load on bridge at } (x, y) \text{ at time } t$$

- General quantity of interest at time t: total load or other force on segment of bridge:

$$\int_{xy} Z(x, y, t) w(x, y) dxdy$$

- Example w: load in strip across bridge between x_1 and x_2 feet out from south side on central span

$$W(t, x_1, x_2) = \int_{x_1}^{x_2} \int_0^{W_B} Z(x, y, t) dydx$$

Quantity of concern to engineers

- Quantity of concern to engineers:

$$M_T(L) \equiv \max_{t\in[0,T]} \max_{0\le x_1\le L_B-L} W(t, x_1, x_1+L)$$

- First modelling assumption. Years $1,\ldots,T$ are iid.
- So:

$$P(M_T(L)\le y) = P(M_1(L)\le y)^T$$

- So: years to first exceedance of level y has geometric distribution with probability of success

$$P(M_1(L) > y)$$

- Find y so this last is $1/100$; expected value of geometric is 100.
- Call this the 100 year return time load.

Next modelling consideration.

- Two kinds of loads: static and dynamic.
- Consider only static not dynamic loading.
- Observation: static loading much higher when traffic stopped than not.
- So: define N to be number of traffic stoppages in year.
- Let $M_{1,n}(L)$ be worst load over segment of length L during nth of N stoppages.
- Idea

$$P(M_1(L) > y) = P(\max_{1\le n\le N} M_{1,n}(L) > y)$$

Treat $M_{1,n}(L)$ as iid *given* N.

Conditioning

- Next: Evaluate $P(M_1(L) > y)$ by conditioning.
- Shorten notation:

$$M = \max_{1 \le n \le N} X_n$$

where X_i iid, cdf F, survival ftn $S = 1 - F$.

$$\begin{aligned} P(M \le y) &= \mathrm{E}\{P(M \le y|N)\} \\ &= \mathrm{E}[\{1 - S(y)\}^N)] \\ &= \phi[\log\{1 - S(y)\}] \end{aligned}$$

where

$$\phi(t) = \mathrm{E}\left(e^{tN}\right)$$

is the moment generating function of N.

Solve for y

- Comment: ϕ is monotone increasing.
- So: if $S(y) \le g(y)$ then

$$P(M > y) \le 1 - \phi[\log\{1 - g(y)\}]$$

  and solving

$$\phi[\log\{1 - g(y)\}] = 0.99$$

  gives larger solution than

$$P(M \le y) = 0.99$$

- Remaining steps:
  1. Model for N.
  2. Model / upper bound for S.

Modelling N:

- Simplest idea: Poisson process of accidents.
- So N has Poisson(λ) dist for some λ.
- Then

$$\phi(t) = \sum e^{-\lambda} \frac{(\lambda e^t)^n}{n!}$$

  which is

$$\phi(t) = \exp\{\lambda(e^t - 1)\}$$

- Criticisms: No allowance for variation in traffic densities, weather, etc from year to year.

More sophisticated assumption

- Potentially better assumption.
- N is overdispersed Poisson, say, Negative Binomial:

$$P(N = k) = \binom{r + k - 1}{k} p^r (1 - p)^k \quad k = 0, 1, \cdots$$

- This makes

$$\mathrm{E}(e^{tX}) = \frac{p^r}{\{1 - (1 - p)e^t\}^r}$$

- Idea: for Poisson $\sigma = \sqrt{\mu}$.
- For Negative Binomial $\mu = r(1 - p)/p$ and

$$\sigma = \sqrt{r(1 - p)/p^2} = \sqrt{1/p}\sqrt{\mu} > \sqrt{\mu}$$

- Idea: use of overdispersed variable makes for longer tails relative to mean.

Use of Upper Bounds

- Now we need to model / bound the survival function S.
- Stoppage lasts random time T.
- During that time traffic builds up behind stoppage; cars jam together.
- Worst section of length L found by sliding window along line of stopped cars to find maximum.
- Notional model (not the way we did it):
- Model vehicles arriving at end of queue.
- Might use Poisson Process.
- Each vehicle has random mass, length, distribution of load along length.
- Random gaps between vehicles.
- Just before traffic starts to move again: look for heaviest segment of length L in stoppage.

Problems

1. different kinds of stoppage: # lanes, direction of flow, location on bridge, cars trickle past?
2. hard to deal with supremum over all segments of length L.
3. specify joint law of mass, length, distribution of load along single vehicle.

- Digression to method we didn't use:
- Model length of stoppage T with density g.
- Model N, number of vehicles arriving at end of stoppage, given T as Poisson(λT).
- Assume next vehicle arriving picked at random; joint density $h(w, l)$ of weight, length. W_i, L_i values for ith arrival.

Physical Length of stoppage

- Assume load distributed evenly along length of vehicle.
- Final length of line at end of stoppage is

$$L_T \equiv \sum_{i=1}^{N} L_i .$$

- Can compute mean, variance, generating function of L_T?

$$\begin{aligned} \mathrm{E}\{\exp(sL)\} &= \mathrm{E}\left[\mathrm{E}\{\exp(sL)|N\}\right] \\ &= \mathrm{E}\left(\left[\mathrm{E}\{\exp(sL_i)\}\right]^N\right) \\ &= \phi_N[\log\{\phi_L(s)\}] \end{aligned}$$

Here each ϕ is a moment generating function.

Compound Poisson processes

- Method of analysis for a compound Poisson Process.
- Can use the mgf of L to compute distribution of L by inversion of Laplace transform.
- Problem: how to scan for maximum load?

Discretization

- Simplify problem: discretize and bound.
- If h is small and $X(s)$ some process then

$$\sup_{0 \le t \le T-\tau} \int_t^{t+\tau} X(s)ds$$

  is close to

$$\max_k \int_{kh}^{kh+\tau} X(s)ds$$

- We took h to be 50 feet and considered $\tau = 50n$ feet.
- Switch from thinking about length of stoppage in time to length of stoppage in multiples of h.
- Let N_i be number of segments of length h building up on bridge during stoppage
- Let $X_j; j = 1, \ldots, N_i$ be the loads on the consecutive segments.

Upper Bounds

- So: our interest is in

$$\max\{X_r + \cdots X_{r+n-1}; 1 \le r \le N_i + 1 - n\}$$

- Upper bound on survival function?
- Define

$$U_r = X_r + \cdots X_{r+n-1}$$

- Argue that

$$1 - P(\max_{1\le r\le m} U_r > y) \le 1 - \prod_{1\le r\le m} \{1 - P(U_r > y)\}$$

  for any m.
- Rationale: Values of U_r are *positive orthant dependent*.
- (Large values of one U suggest large values of adjacent U.)

Modelling Details

- So now:

$$P(\max\{U_r; 1 \le r \le N_i\} > y) \le 1 - \phi_{N_i}[\log\{1 - S_U(y)\}]$$

- Model law of X_i by considering possible loading patterns by cars, trucks, buses.
- We took cars to be fixed length and weight.
- Same for buses.
- Trucks had fixed length, weight uniform on 12 to 40 tons.
- Computed moment generating function of an X:

$$\phi_X(t) = \mathrm{E}(e^{tX})$$

Markov's inequality

- Final step. Need to compute S_U.
- Instead use Markov's inequality:

$$P(X \geq x) \leq \frac{\mathrm{E}\{g(X)\}}{g(x)}$$

for any increasing positive g.

- Choose $g(\cdot) = \exp(h\cdot)$.
- So:

$$\begin{aligned} S_U(y) &\leq \frac{\mathrm{E}(e^{hU})}{\exp(hy)} \\ &= \frac{\{\phi_X(h)\}^n}{\exp(hy)} \end{aligned}$$

where

$$\phi_X(h) = \mathrm{E}\{\exp(hX)\}$$

Optimize Markov's inequality

- These can be assembled to give a bound depending on h.
- Then: minimize over $h > 0$ to find good bound.
- I wrote FORTRAN code to do this in 1975 for my summer job with Jim Zidek.
- He was consulting with Frank Navin for Peter Buckland.
- Zidek, James V., Navin, Francis P. D. and Lockhart, R. A. (1979). Statistics of extremes: An alternate method with application to bridge design codes. *Technometrics*, **21**, 185–191.

Summary of Today's Lecture

- Notice big picture of modelling process.
- Define variables, introduce notation.
- Formulate problem (here as one of bounding return period).
- Make assumptions about joint and conditional distributions.
- Use conditioning.
- Use independence, overdispersion, dependence notions.
- Use Markov's inequality or numerical inversion of Laplace transform.
- Notice wealth of detailed modelling assumptions