Poisson Processes

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Purposes of Today's Lecture

- Give 3 definitions of Poisson Process.
- Show definitions are equivalent.
- Describe processes arising from generalization of approaches.

Poisson Processes

- Particles arriving over time at a particle detector.
- Several ways to describe most common model.
- Approach 1:
  - **a)** numbers of particles arriving in an interval has Poisson distribution,
  - **b)** mean proportional to length of interval,
  - **c)** numbers in several non-overlapping intervals independent.
- For $s < t$, denote number of arrivals in $(s, t]$ by $N(s, t)$.
- Jargon: $N(A) =$ number of points in A is a **counting process**.
- $N(s, t)$ has a Poisson$(\lambda(t - s))$ distribution.
- For $0 \le s_1 < t_1 \le s_2 < t_2 \cdots \le s_k < t_k$ the variables $N(s_i, t_i); i = 1, \ldots, k$ are independent.

Poisson Processes: Approach 2

- Let $0 < S_1 < S_2 < \cdots$ be the times at which the particles arrive.
- Let $T_i = S_i - S_{i-1}$ with $S_0 = 0$ by convention.
- T_i are called **interarrival** times.
- Then $T_1, T_2, \ldots$ are independent Exponential random variables with mean $1/\lambda$.
- Note $P(T_i > x) = e^{-\lambda x}$ is called **survival function** of T_i.
- Approaches are equivalent.
- Both are deductions of a model based on **local** behaviour of process.

Poisson Processes: Approach 3

Assume:

1. given all the points in $[0, t]$ the probability of 1 point in the interval $(t, t+h]$ is of the form
$$\lambda h + o(h)$$
2. given all the points in $[0, t]$ the probability of 2 or more points in interval $(t, t+h]$ is of the form
$$o(h)$$

Notation: given functions f and g we write

$$f(h) = g(h) + o(h)$$

provided

$$\lim_{h \to 0} \frac{f(h) - g(h)}{h} = 0$$

Landau notation

[Aside: if there is a constant M such that

$$\limsup_{h \to 0} \left| \frac{f(h) - g(h)}{h} \right| \le M$$

we say

$$f(h) = g(h) + O(h)$$

Notation due to Landau. Another form is

$$f(h) = g(h) + O(h)$$

means there is $\delta > 0$ and M s.t. for all $|h| < \delta$

$$|f(h) - g(h)| \le M|h|$$

Idea: $o(h)$ is tiny compared to h while $O(h)$ is (very) roughly the same size as h.]

Generalizations of Poisson Processes

1. First (Poisson) model generalizes to $N(s,t]$ having a Poisson distribution with parameter $\Lambda(t) - \Lambda(s)$ for some non-decreasing non-negative function Λ (called **cumulative intensity**). Result called **inhomogeneous** Poisson process.
2. Exponential interarrival model generalizes to independent non-exponential interarrival times. Result is **renewal process** or **semi-Markov** process.
3. Infinitesimal probability model generalizes to other infinitesimal jump rates. Model specifies **infinitesimal generator**. Yields other **continuous time Markov Chains**.

Equivalence of Modelling Approaches

- I show: 3 implies 1, 1 implies 2 and 2 implies 3. First explain o, O.
- Model 3 implies 1: Fix t, define $f_t(s)$ to be conditional probability of 0 points in $(t, t+s]$ given value of process on $[0, t]$.
- Derive differential equation for f.
- Given process on $[0, t]$ and 0 points in $(t, t+s]$ probability of no points in $(t, t+s+h]$ is

$$f_{t+s}(h) = 1 - \lambda h + o(h)$$

Equivalence of Modelling Approaches

Given the process on $[0, t]$ the probability of no points in $(t, t+s]$ is $f_t(s)$. Using $P(AB|C) = P(A|BC)P(B|C)$ gives

$$\begin{aligned} f_t(s+h) &= f_t(s) f_{t+s}(h) \\ &= f_t(s)(1 - \lambda h + o(h)) \end{aligned}$$

Now rearrange, divide by h to get

$$\frac{f_t(s+h) - f_t(s)}{h} = -\lambda f_t(s) + \frac{o(h)}{h}$$

Let $h \to 0$ and find

$$\frac{\partial f_t(s)}{\partial s} = -\lambda f_t(s)$$

Differential equation has solution

$$f_t(s) = f_t(0) \exp(-\lambda s) = \exp(-\lambda s)\,.$$

Equivalence of Modelling Approaches

Things to notice:

- $f_t(s) = e^{-\lambda s}$ is survival function of exponential rv..
- We had suppressed dependence of $f_t(s)$ on $N(u); 0 \le u \le t$ but solution does not depend on condition.
- So: the event of getting 0 points in $(t, t+s]$ is independent of $N(u); 0 \le u \le t$.
- We used: $f_t(s)o(h) = o(h)$. Other rules:

$$\begin{aligned} o(h) + o(h) &= o(h) \\ O(h) + O(h) &= O(h) \\ O(h) + o(h) &= O(h) \\ o(h^r)O(h^s) &= o(h^{r+s}) \\ O(o(h)) &= o(h) \end{aligned}$$

Equivalence of Modelling Approaches

General case: notation: $N(t) = N(0,t)$.
$N(t)$ is a non-decreasing function of t. Let

$$P_k(t) = P(N(t) = k)$$

Evaluate $P_k(t+h)$: condition on $N(s); 0 \le s < t$ and on $N(t) = j$.
Given $N(t) = j$ probability that $N(t+h) = k$ is conditional probability of $k - j$ points in $(t, t+h]$.
So, for $j \le k-2$:

$$P(N(t+h) = k | N(t) = j, N(s), 0 \le s < t) = o(h)$$

For $j = k-1$ we have

$$P(N(t+h) = k | N(t) = k-1, N(s), 0 \le s < t) = \lambda h + o(h)$$

For $j = k$ we have

$$P(N(t+h) = k | N(t) = k, N(s), 0 \le s < t) = 1 - \lambda h + o(h)$$

Equivalence of Modelling Approaches

N is increasing so only consider $j \le k$.

$$\begin{aligned} P_k(t+h) &= \sum_{j=0}^{k} P(N(t+h)=k|N(t)=j)P_j(t) \\ &= P_k(t)(1-\lambda h) + \lambda h P_{k-1}(t) + o(h) \end{aligned}$$

Rearrange, divide by h and let $h \to 0$ t get

$$P_k'(t) = -\lambda P_k(t) + \lambda P_{k-1}(t)$$

For $k = 0$ the term P_{k-1} is dropped and

$$P_0'(t) = -\lambda P_0(t)$$

Using $P_0(0) = 1$ we get

$$P_0(t) = e^{-\lambda t}$$

Equivalence of Modelling Approaches

Put this into the equation for $k = 1$ to get

$$P_1'(t) = -\lambda P_1(t) + \lambda e^{-\lambda t}$$

Multiply by $e^{\lambda t}$ to see

$$\left(e^{\lambda t} P_1(t)\right)' = \lambda$$

With $P_1(0) = 0$ we get

$$P_1(t) = \lambda t e^{-\lambda t}$$

For general k we have $P_k(0) = 0$ and

$$\left(e^{\lambda t} P_k(t)\right)' = \lambda e^{\lambda t} P_{k-1}(t)$$

Check by induction that

$$e^{\lambda t} P_k(t) = (\lambda t)^k / k!$$

Hence: $N(t)$ has Poisson(λt) distribution.

Extension

Similar ideas permit proof of

$$P(N(s,t) = k | N(u); 0 \le u \le s) = \frac{\{\lambda(t-s)\}^k e^{-\lambda}}{k!}$$

Now prove (by induction) N has independent Poisson increments.

Exponential Interarrival Times

- If N is a Poisson Process we define $T_1, T_2, \ldots$ to be the times between 0 and the first point, the first point and the second and so on.
- Fact: $T_1, T_2, \ldots$ are iid exponential rvs with mean $1/\lambda$.
- We already did T_1 rigorously.
- The event $T_1 > t$ is exactly the event $N(t) = 0$.
- So

$$P(T_1 > t) = \exp(-\lambda t)$$

which is the survival function of an exponential rv.

Exponential Interarrival Times

- Do case of T_1, T_2.
- Let t_1, t_2 be two positive numbers and $s_1 = t_1$, $s_2 = t_1 + t_2$.
- Event

$$\{t_1 < T_1 \le t_1 + \delta_1\} \cap \{t_2 < T_2 \le t_2 + \delta_2\}.$$

is almost the same as the intersection of four events:

$$\begin{aligned} N(0, t_1] &= 0 \\ N(t_1, t_1 + \delta_1] &= 1 \\ N(t_1 + \delta_1, t_1 + \delta_1 + t_2] &= 0 \\ N(s_2 + \delta_1, s_2 + \delta_1 + \delta_2] &= 1 \end{aligned}$$

which has probability

$$e^{-\lambda t_1} \times \lambda\delta_1 e^{-\lambda\delta_1} \times e^{-\lambda t_2} \times \lambda\delta_2 e^{-\lambda\delta_2}$$

Divide by $\delta_1\delta_2$, let δ_1, δ_2 go to 0 to get joint density of T_1, T_2 is

$$\lambda^2 e^{-\lambda t_1} e^{-\lambda t_2}$$

which is the joint density of two independent exponential variates.

More rigor

- Find joint density of $S_1, \ldots, S_k$.
- Use **change of variables** to find joint density of $T_1, \ldots, T_k$.

First step: Compute

$$P(0 < S_1 \le s_1 < S_2 \le s_2 \cdots < S_k \le s_k)$$

This is just the event of exactly 1 point in each interval $(s_{i-1}, s_i]$ for $i = 1, \ldots, k-1$ $(s_0 = 0)$ and at least one point in $(s_{k-1}, s_k]$ which has probability

$$\prod_1^{k-1} \left\{ \lambda(s_i - s_{i-1}) e^{-\lambda(s_i - s_{i-1})} \right\} \left(1 - e^{-\lambda(s_k - s_{k-1})} \right)$$

Second step

Write this in terms of joint cdf of $S_1, \ldots, S_k$. I do $k = 2$:

$$P(0 < S_1 \le s_1 < S_2 \le s_2) = F_{S_1,S_2}(s_1, s_2) - F_{S_1,S_2}(s_1, s_1)$$

Notice tacit assumption $s_1 < s_2$. Differentiate twice, that is, take

$$\frac{\partial^2}{\partial s_1 \partial s_2}$$

to get

$$f_{S_1,S_2}(s_1, s_2) = \frac{\partial^2}{\partial s_1 \partial s_2} \lambda s_1 e^{-\lambda s_1} \left(1 - e^{-\lambda(s_2 - s_1)}\right)$$

Simplify to

$$\lambda^2 e^{-\lambda s_2}$$

Recall tacit assumption to get

$$f_{S_1,S_2}(s_1, s_2) = \lambda^2 e^{-\lambda s_2} 1(0 < s_1 < s_2)$$

That completes the first part.

Joint cdf to joint density

Now compute the joint cdf of T_1, T_2 by

$$F_{T_1,T_2}(t_1,t_2) = P(S_1 < t_1, S_2 - S_1 < t_2)$$

This is

$$\begin{aligned} P(S_1 < t_1, S_2 - S_1 < t_2) &= \int_0^{t_1} \int_{s_1}^{s_1+t_2} \lambda^2 e^{-\lambda s_2} ds_2 ds_1 \\ &= \lambda \int_0^{t_1} \left(e^{-\lambda s_1} - e^{-\lambda(s_1+t_2)} \right) ds_1 \\ &= 1 - e^{-\lambda t_1} - e^{-\lambda t_2} + e^{-\lambda(t_1+t_2)} \end{aligned}$$

Differentiate twice to get

$$f_{T_1,T_2}(t_1,t_2) = \lambda e^{-\lambda t_1} \lambda e^{-\lambda t_2}$$

which is the joint density of two independent exponential random variables.

Summary so far

Have shown:

- Instantaneous rates model implies independent Poisson increments model implies independent exponential interarrivals.
- Next: show independent exponential interarrivals implies the instantaneous rates model.
- Suppose $T_1, \ldots$ iid exponential rvs with means $1/\lambda$. Define N_t by $N_t = k$ if and only if

$$T_1 + \cdots + T_k \le t \le T_1 + \cdots + T_{k+1}$$

- Let A be event $N(s) = n(s); 0 < s \le t$. We are to show

$$P(N(t, t+h] = 1 | N(t) = k, A) = \lambda h + o(h)$$

and

$$P(N(t, t+h] \ge 2 | N(t) = k, A) = o(h)$$

Markov Property

If $n(s)$ is a possible trajectory consistent with $N(t) = k$ then n has jumps at points

$$t_1, t_1 + t_2, \ldots, s_k \equiv t_1 + \cdots + t_k < t$$

and at no other points in $(0, t]$.
So given $N(s) = n(s); 0 < s \leq t$ with $n(t) = k$ we are essentially being given

$$T_1 = t_1, \ldots, T_k = t_k, T_{k+1} > t - s_k$$

and asked the conditional probability in the first case of the event B given by

$$t - s_k < T_{k+1} \leq t - s_k + h < T_{k+2} + T_{k+1} \, .$$

Conditioning on $T_1, \ldots, T_k$ irrelevant (independence).

Markov Property

$$P(N(t,t+h] = 1 | N(t) = k, A)/h = P(B | T_{k+1} > t - s_k)/h = \frac{P(B)}{he^{-\lambda(t-s_k)}}$$

Numerator evaluated by integration:

$$P(B) = \int_{t-s_k}^{t-s_k+h} \int_{t-s_k+h-u_1}^{\infty} \lambda^2 e^{-\lambda(u_1+u_2)} du_2 du_1$$

Let $h \to 0$ to get the limit

$$P(N(t,t+h] = 1 | N(t) = k, A)/h \to \lambda$$

as required.
The computation of

$$\lim_{h \to 0} P(N(t,t+h] \geq 2 | N(t) = k, A)/h$$

is similar.

Properties of exponential rvs

Convolution: If X and Y independent rvs with densities f and g respectively and $Z = X + Y$ then

$$P(Z \le z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f(x) g(y) dy dx$$

Differentiating wrt z we get

$$f_Z(z) = \int_{-\infty}^{\infty} f(x) g(z-x) dx$$

This integral is called the **convolution** of densities f and g.
If $T_1, \ldots, T_n$ iid Exponential(λ) then $S_n = T_1 + \cdots + T_n$ has a Gamma(n, λ) distribution. Density of S_n is

$$f_{S_n}(s) = \lambda (\lambda s)^{n-1} e^{-\lambda s} / (n-1)!$$

for $s > 0$.

Convolution Property Proved

Proof:

$$P(S_n > s) = P(N(0,s] < n) = \sum_{j=0}^{n-1} (\lambda s)^j e^{-\lambda s}/j!$$

Then

$$\begin{aligned}
f_{S_n}(s) &= \frac{d}{ds} P(S_n \le s) = \frac{d}{ds}\{1 - P(S_n > s)\} \\
&= -\lambda \sum_{j=1}^{n-1} \left\{ j(\lambda s)^{j-1} - (\lambda s)^j \right\} \frac{e^{-\lambda s}}{j!} + \lambda e^{-\lambda s} \\
&= \lambda e^{-\lambda s} \sum_{j=1}^{n-1} \left\{ \frac{(\lambda s)^j}{j!} - \frac{(\lambda s)^{j-1}}{(j-1)!} \right\} + \lambda e^{-\lambda s}
\end{aligned}$$

This telescopes to

$$f_{S_n}(s) = \lambda (\lambda s)^{n-1} e^{-\lambda s}/(n-1)!$$

Extreme Values

Extreme Values: If $X_1, \ldots, X_n$ are independent exponential rvs with means $1/\lambda_1, \ldots, 1/\lambda_n$ then $Y = \min\{X_1, \ldots, X_n\}$ has an exponential distribution with mean

$$\frac{1}{\lambda_1 + \cdots + \lambda_n}$$

Proof:

$$\begin{aligned} P(Y > y) &= P(\forall k X_k > y) \\ &= \prod e^{-\lambda_k y} \\ &= e^{-\sum \lambda_k y} \end{aligned}$$

Memoryless property

Memoryless Property: conditional distribution of $X - x$ given $X \geq x$ is exponential if X has an exponential distribution.
Proof:

$$
\begin{aligned}
P(X - x > y | X \geq x) &= \frac{P(X > x + y, X \geq x)}{P(X > x)} \\
&= \frac{PX > x + y)}{P(X \geq x)} \\
&= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} \\
&= e^{-\lambda y}
\end{aligned}
$$

Hazard Rates

The hazard rate, or instantaneous failure rate for a positive random variable T with density f and cdf F is

$$r(t) = \lim_{\delta \to 0} \frac{P(t < T \le t + \delta | T \ge t)}{\delta}$$

This is just

$$r(t) = \frac{f(t)}{1 - F(t)}$$

For an exponential random variable with mean $1/\lambda$ this is

$$h(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

The exponential distribution has constant failure rate.

Weibull variates

Weibull random variables have density

$$f(t|\lambda,\alpha) = \lambda(\lambda t)^{\alpha-1} e^{-(\lambda t)^\alpha}$$

for $t > 0$. The corresponding survival function is

$$1 - F(t) = e^{-(\lambda t)^\alpha}$$

and the hazard rate is

$$r(t) = \lambda(\lambda t)^{\alpha-1};$$

increasing for $\alpha > 1$, decreasing for $\alpha < 1$. For $\alpha = 1$: exponential distribution.
Since

$$r(t) = \frac{dF(t)/dt}{1-F(t)} = -\frac{d\log(1-F(t))}{dt}$$

we can integrate to find

$$1 - F(t) = \exp\{-\int_0^t r(s)ds\}$$

so that r determines F and f.

Properties of Poisson Processes

1) If N_1 and N_2 are independent Poisson processes with rates λ_1 and λ_2, respectively, then $N = N_1 + N_2$ is a Poisson processes with rate $\lambda_1 + \lambda_2$.
2) Let N be a Poisson process with rate λ.
• Suppose each point is marked with a label, say one of $L_1, \ldots, L_r$, independently of all other occurrences.
• Suppose p_i is the probability that a given point receives label L_i.
• Let N_i count the points with label i (so that $N = N_1 + \cdots + N_r$).
• Then $N_1, \ldots, N_r$ are independent Poisson processes with rates $p_i\lambda$.

Properties 3, 4 and 5

3) Suppose $U_1, U_2, \ldots$ independent rvs, each uniformly distributed on $[0, T]$.

• Suppose M is a Poisson(λT) random variable independent of the U's.

• Let

$$N(t) = \sum_1^M 1(U_i \le t)$$

• Then N is a Poisson process on $[0, T]$ with rate λ.

4) Let N be Poisson process with rate λ.

• $S_1 < S_2 < \cdots$ times at which points arrive

• Given $N(T) = n$, $S_1, \ldots, S_n$ have same distribution as order statistics of sample of size n from uniform distribution on $[0, T]$.

5) Given $S_{n+1} = T$, $S_1, \ldots, S_n$ have same distribution as order statistics of sample of size n from uniform distribution on $[0, T]$.

Indications of some proofs: **1**

- $N_1, \ldots, N_r$ independent Poisson processes rates λ_i, $N = \sum N_i$.
- Let A_h be the event of 2 or more points in N in the time interval $(t, t+h]$, B_h, the event of exactly one point in N in the time interval $(t, t+h]$.
- Let A_{ih} and B_{ih} be the corresponding events for N_i.
- Let H_t denote the history of the processes up to time t; we condition on H_t.
- Technically, H_t is the σ-field generated by

$$\{N_i(s); 0 \le s \le t, i = 1, \ldots, r\}$$

Proof of **1** continued

We are given:

$$P(A_{ih}|H_t) = o(h)$$

and

$$P(B_{ih}|H_t) = \lambda_i h + o(h)\,.$$

Note that

$$A_h \subset \bigcup_{i=1}^{r} A_{ih} \cup \bigcup_{i \neq j} (B_{ih} \cap B_{jh})$$

Proof of **1** continued

Since

$$\begin{aligned} P(B_{ih} \cap B_{jh} | H_t) &= P(B_{ih}|H_t)P(B_{jh}|H_t) \\ &= (\lambda_i h + o(h))(\lambda_j h + o(h)) \\ &= O(h^2) \\ &= o(h) \end{aligned}$$

and

$$P(A_{ih}|H_t) = o(h)$$

we have checked one of the two infinitesimal conditions for a Poisson process.

Proof of 1 continued

Next let C_h be the event of no points in N in the time interval $(t, t+h]$ and C_{ih} the same for N_i. Then

$$\begin{aligned} P(C_h|H_t) &= P(\cap C_{ih}|H_t) \\ &= \prod P(C_{ih}|H_t) \\ &= \prod (1 - \lambda_i h + o(h)) \\ &= 1 - (\sum \lambda_i)h + o(h) \end{aligned}$$

shows

$$\begin{aligned} P(B_h|H_t) &= 1 - P(C_h|H_t) - P(A_h|H_t) \\ &= (\sum \lambda_i)h + o(h) \end{aligned}$$

Hence N is a Poisson process with rate $\sum \lambda_i$.

Proof of **2**

- The infinitesimal approach used for 1 can do part of this.
- See Ross for rest.
- Events defined as in **1)**
  - B_{ih} — there is one point in N_i in $(t, t+h]$ is the event
  - B_h — there is exactly one point in any of the r processes together with a subset of A_h where there are two or more points in N in $(t, t+h]$ but exactly one is labeled i.
- Since $P(A_h|H_t) = o(h)$

$$\begin{aligned} P(B_{ih}|H_t) &= p_i P(B_h|H_t) + o(h) \\ &= p_i(\lambda h + o(h)) + o(h) \\ &= p_i \lambda h + o(h) \end{aligned}$$

- Similarly, A_{ih} is a subset of A_h so

$$P(A_{ih}|H_t) = o(h)$$

- This shows each N_i is Poisson with rate λp_i.
- Independence is more work; see homework for easier algebraic method.

Proof of **3**

- Fix $s < t$.
- Let $N(s,t)$ be number of points in $(s,t]$.
- Given $M = n$ conditional dist of $N(s,t)$ is Binomial(n,p) with $p = (t-s)/T$.
- So

$$
\begin{aligned}
P(N(s,t)=k) &= \sum_{n=k}^{\infty} P(N(s,t)=k, N=n) \\
&= \sum_{n=k}^{\infty} P(N(s,t)=k | N=n) P(N=n) \\
&= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \frac{(\lambda T)^n}{n!} e^{-\lambda T}
\end{aligned}
$$

Proof of **3** continued

$$
\begin{aligned}
P(N(s,t)=k) &= \frac{e^{-\lambda T}}{k!}(\lambda T p)^k \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}(\lambda T)^{n-k}}{(n-k)!} \\
&= \frac{e^{-\lambda T}}{k!}(\lambda T p)^k \sum_{m=0}^{\infty} (1-p)^m(\lambda T)^m/m! \\
&= \frac{e^{-\lambda T}}{k!}(\lambda T p)^k e^{-\lambda T(1-p)} \\
&= \frac{e^{-\lambda(t-s)}(\lambda(t-s))^k}{k!}
\end{aligned}
$$

Proof of **4**

Fix s_i, h_i for $i = 1, \ldots, n$ such that

$$0 < s_1 < s_1 + h_1 < s_2 < \cdots < s_n < s_n + h_n < T$$

Given $N(T) = n$ we compute the probability of the event

$$A = \bigcap_{i=1}^{n} \{s_i < S_i < s_i + h_i\}$$

Intersection of A,1 $N(T) = n$ is ($s_0 = h_0 = 0$):

$$B \equiv \bigcap_{i=1}^{n} \{N(s_{i-1} + h_{i-1}, s_i] = 0, N(s_i, s_i + h_i] = 1\} \cap \{N(s_n + h_n, T] = 0\}$$

whose probability is

$$\left(\prod \lambda h_i\right) e^{-\lambda T}$$

Proof of **4** continued

So

$$
\begin{aligned}
P(A|N(t)=n) &= \frac{P(A, N(T)=n)}{(N(T)=n)} \\
&= \frac{\lambda^n e^{-\lambda T} \prod h_i}{(\lambda T)^n e^{-\lambda T}/n!} \\
&= \frac{n! \prod h_i}{T^n}
\end{aligned}
$$

Divide by $\prod h_i$ and let all h_i go to 0 to get joint density of $S_1, \ldots, S_n$ is

$$
\frac{n!}{T^n} 1(0 < s_1 < \cdots < s_n < T)
$$

which is the density of order statistics from a Uniform$[0, T]$ sample of size n.

Proof of **5**

Replace the event $S_{n+1} = T$ with $T < S_{n+1} < T + h$. With A as before we want

$$P(A | T < S_{n+1} < T + h) = \frac{P(B, N(T, T+h] \geq 1)}{P(T < S_{n+1} < T + h)}$$

Note that B is independent of $\{N(T, T+h] \geq 1\}$ and that we have already found the limit

$$\frac{P(B)}{\prod h_i} \to \lambda^n e^{-\lambda T}$$

Proof of **5** continued

We are left to compute the limit of

$$\frac{P(N(T,T+h] \geq 1)}{P(T < S_{n+1} < T+h)}$$

The denominator is

$$F_{S_{n+1}}(t+h) - F_{S_{n+1}}(t) = f_{S_{n+1}}(t)h + o(h)$$

Thus

$$\begin{aligned}\frac{P(N(T,T+h] \geq 1)}{P(T < S_{n+1} < T+h)} &= \frac{\lambda h + o(h)}{\frac{(\lambda T)^n}{n!}e^{-\lambda T}\lambda h + o(h)} \\ &\to \frac{n!}{(\lambda T)^n e^{-\lambda T}}\end{aligned}$$

This gives the conditional density of $S_1, \ldots, S_n$ given $S_{n+1} = T$ as in **4)**.

Inhomogeneous Poisson Processes

Hazard rate can be used to extend notion of Poisson Process.
Suppose $\lambda(t) \geq 0$ is a function of t.
Suppose N is a counting process such that

$$P(N(t+h) = k+1 | N(t) = k, H_t) = \lambda(t)h + o(h)$$

and

$$P(N(t+h) \geq k+2 | N(t) = k, H_t) = o(h)$$

Then:
a) N has independent increments and
b) $N(t+s) - N(t)$ has Poisson distribution with mean

$$\int_t^{t+s} \lambda(u) du$$

Inhomogeneous Poisson Processes

If we put

$$\Lambda(t) = \int_0^t \lambda(u) du$$

then mean of $N(t+s) - N(t)$ is $\Lambda(t+s) - \Lambda(t)$.
Jargon: λ is the **intensity** or **instantaneous intensity** and Λ the **cumulative intensity**.
Can use the model with Λ any non-decreasing right continuous function, possibly without a derivative. This allows ties.

Space Time Poisson Processes

- Suppose at each time S_i of a Poisson Process, N, we have rv Y_i with the Y_i iid and independent of the Poisson process.
- Let M be the counting process on $[0,\infty) \times \mathcal{Y}$ (where $\mathcal{Y}$ is the range space of the Ys) defined by

$$M(A) = \#\{(S_i, Y_i) \in A\}$$

  Then M is an inhomogeneous Poisson process with mean function μ a measure extending

$$\mu([a,b] \times C) = \lambda(b-a)P(Y \in C)$$

- This means that each $M(A)$ has a Poisson distribution with mean $\mu(A)$ and if $A_1, \ldots, A_r$ are disjoint then $M(A_1), \ldots, M(A_r)$ are independent.
- The proof in general is a monotone class argument.
- The first step is: if $(a_i, b_i), i = 1, \ldots, r$ are disjoint intervals and $C_1, \ldots, C_s$ disjoint subsets of $\mathcal{Y}$ then the rs rvs $M((a_i, b_i) \times C_j)$ are independent Poisson random variables.
- See the homework for proof of a special case.