

# Stochastic Differential Equations

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# Purposes of Today's Lecture

- Motivate Stochastic Differential Equations.
- Describe Ito and Stratonovich integrals.



# Stochastic Differential Equations

- Return to definition of diffusion: given  $\mathcal{H}_t$

$$X(t+h) = X(t) + \mu(X(t))h + \sigma(X(t))\sqrt{h}\epsilon + o(h)$$

where  $\epsilon \sim N(0, 1)$ ;  $\mu(\cdot)$  and  $\sigma(\cdot)$  are model specified functions.

- Use Brownian motion to give  $\epsilon$ :

$$\begin{aligned} X(t+h) &= X(t) + \mu(X(t))h \\ &\quad + \sigma(X(t))\{B(t+h) - B(t)\} + o(h) \end{aligned}$$

- Usually written in differential form  $h = dt$ :

$$dX_t = \mu(X(t))dt + \sigma(X(t))dB_t$$

- Interpretation is integral:

$$X_t = X_0 + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dB_s$$

- Meaning?



# Riemann-Stieltjes Integrals

- $F$  monotone increasing (right continuous, say) on  $[0, t]$ ;  $f$  continuous on  $[0, t]$ :

$$\int_0^t f(s)F(ds)$$

is defined as follows.

- Fix  $N$ . Let  $t_k = tk/N$  for  $k = 0, 1, \dots, N$ .
- Put

$$\bar{I}_N = \sum_{k=1}^n \max\{f(u) : t_{k-1} \leq u \leq t_k\} \{F(t_k) - F(t_{k-1})\}$$

and

$$\underline{I}_N = \sum_{k=1}^n \min\{f(u) : t_{k-1} \leq u \leq t_k\} \{F(t_k) - F(t_{k-1})\}$$



## Riemann-Stieltjes integrals continued

- Then

$$\int_0^t f(s)F(ds) = \lim_N \bar{T}_N = \lim_N \underline{L}_N$$

- If  $F$  is absolutely continuous then

$$\int_0^t f(s)F(ds) = \int_0^t f(s)F'(s)ds$$

which is an ordinary Riemann integral.



# The integration problem

- Idea extends to  $F$  of "bounded variation" – difference of two monotone increasing  $G$ .
- Back to SDE: What is

$$\int_0^t \sigma(X(s)) dB_s?$$

- Problem: bounded variation means

$$\sup_N \sum_k |F(t_k) - F(t_{k-1})| < \infty$$

- But

$$\lim_{N \rightarrow \infty} \sum_k |B(t_k) - B(t_{k-1})| = \infty$$

because these are sums of  $N$  iid terms with mean proportional to  $1/\sqrt{N}$ .

- Consider example to see details of problem.



# Stochastic Integrals

- What is

$$\int_0^t B_s dB_s?$$

NOT

$$B_t^2/2$$

- Two discrete approximations in spirit of Riemann Stieltjes:

$$I_{1,N} \equiv \sum_{k=0}^{N-1} B(t_k) \{B(t_{k+1}) - B(t_k)\}$$

and

$$I_{2,N} \equiv \sum_{k=0}^{N-1} B(t_{k+1}) \{B(t_{k+1}) - B(t_k)\}$$



# Stochastic Integrals Continued

- First has mean 0.
- Notice

$$I_{2,N} - I_{1,N} = \sum_{k=0}^{N-1} \{B(t_{k+1}) - B(t_k)\}^2$$

- If we multiply the  $k$ th term by  $N$  we get a  $\chi_1^2$  random variable so the difference is an average of  $N$  independent  $\chi_1^2$ s.





# Unbounded variation bounded quadratic variation

- So

$$I_{2,N} - I_{1,N} \rightarrow t$$

On the other hand

$$I_{2,N} + I_{1,N} = B_t^2 - B_0^2 = B_t^2$$

- Thus

$$I_{1,n} \rightarrow (B_t^2 - t)/2 \text{ and } I_{2,n} \rightarrow (B_t^2 + t)/2$$

- Use centered value of  $B$  in definition to make  $B_t^2/2$  appear.
- The Ito integral

$$\int_0^t B_s dB_s = (B_t^2 - t)/2$$

is a match for our modelling tactic above.

- Centred version is Stratonovich integral.



# Questions of interest

- Existence of solutions of SDEs?
- Calculus of stochastic integrals.



## Example 1: Geometric Brownian Motion

- Consider  $\mu(x) = \alpha x$  and  $\sigma(x) = \beta x$  for  $x > 0$ ,  $\beta > 0$ .
- Idea is change in  $X_t$  has mean and standard deviation proportional to  $X_t$ .
- So both constant in percentage terms.
- Solution of

$$dX_t = \alpha X_t dt + \beta X_t dB_t$$

is Geometric Brownian Motion:

$$X_t = X_0 \exp \left\{ (\alpha - \beta^2/2)t + \beta B_t \right\}.$$



## Example 2: Fisher Wright model

- Wright Fisher or Fisher Wright model of mutation.
- Idea is population of  $N$  individuals of genetic type  $A$  or  $a$ .
- Total number of genes is  $2N$ .
- Random pairing to form next generation: number of  $A$  genes has Binomial( $2N, p$ ) distribution where  $p$  is fraction of current generation which is type  $A$ .
- BUT probability individual gene mutates  $A$  to  $a$  is  $\alpha$ .
- AND probability individual gene mutates  $a$  to  $A$  is  $\alpha'$ .
- Get discrete time Markov Chain:  $X_n$  is number of  $A$  in generation  $n$ ; only change is to  $p$  in Binomial law. Given  $X_n = i$

$$p = \frac{i}{2N}(1 - \alpha) + \frac{2N - i}{2N}\alpha'.$$



# Fisher-Wright in Continuous Time

- Mutation rates are  $\alpha = \delta/(2N)$  and  $\alpha' = \delta'/(2N)$ .
- Let  $N \rightarrow \infty$ .
- $X_n(t)$  is proportion of population of type  $A$  at time  $t = n/(2N)$  and  $X_t$  is the limit.
- Get

$$dX_t = \{-\delta X_t + \delta'(1 - X_t)\} dt + \sqrt{X_t(1 - X_t)} dB_t$$

- Fact: solution exists with  $0 < X_t < 1$  for all  $t$
- Fact: solution is Markov process with stationary initial distribution.



## Example 3: Ornstein Uhlenbeck

- Ornstein Uhlenbeck model: velocity of Brownian particle
- Model velocity of particle (not position as in Brownian motion).
- Introduce friction proportional to velocity:

$$dV_t = -\alpha V_t dt + \sigma dB_t.$$

- Solution is

$$V_t = e^{-\alpha t} \left\{ V_0 + \sigma \int_0^t e^{\alpha s} dB_s \right\}.$$

- This is a Gaussian process (joint distributions are normal).
- Its integral gives position.
- The process has a stationary initial distribution.



## Stationary Initial Distributions

- If a stationary initial density  $\pi$  exists then

$$\pi(y) = \lim_{t \rightarrow \infty} f(t, x, y)$$

- In this case we may expect

$$\lim_{t \rightarrow \infty} \frac{\partial}{\partial t} f(t, x, y) = 0.$$

- The Kolmogorov **forward** equation involves this partial derivative.
- Back to Chapman Kolmogorov in general form.
- For Markov process which might not have stationary transitions:  $f(s, t, x, y)$  is conditional density (at  $y$ ) of  $X(t)$  given  $X(s) = x$ .
- Then for  $s < u < t$

$$f(s, t, x, y) = \int f(s, u, x, z) f(u, t, z, y) dz$$



# Kolmogorov Forward Equation

- Replace  $s$  by  $0$ ,  $t$  by  $t + h$  and  $u$  by  $t$ .
- Use fact  $f(t, t + h, z, y)$  is approximately normal density with mean  $\mu(z)h$  and variance  $\sigma^2(z)h$ :

$$\begin{aligned} f(t, t + h, z, y) &\approx \frac{1}{\sigma(z)\sqrt{2\pi h}} \exp\left(-\frac{(y - z - \mu(z)h)^2}{2\sigma^2(z)h}\right) \\ &= \frac{1}{\sigma(z)\sqrt{2\pi h}} \exp\left(-\frac{(z - y + \mu(z)h)^2}{2\sigma^2(z)h}\right) \end{aligned}$$

- Change variables to

$$u = \frac{z - y + \mu(y)h}{\sigma(y)\sqrt{h}}$$

- Notice  $\mu(y), \sigma(y)$ , not  $\mu(z), \sigma(z)$ .





## Kolmogorov Forward Equation Continued

- After substitution expand

$$f(s, t, x, z)f(t, t + h, z, y)$$

in powers of  $\sqrt{h}$ .

- Lengthy algebra ensues, smoke clears (with aid, for me, of Maple) to give

$$\frac{\partial}{\partial t} f(t, x, y) = -\frac{\partial}{\partial y} \mu(y) f(t, x, y) + \frac{1}{2} \frac{\partial^2}{\partial y^2} \sigma^2(y) f(t, x, y)$$

- So stationary density  $\pi$  satisfies

$$\frac{\partial}{\partial y} \mu(y) \pi(y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} \sigma^2(y) \pi(y)$$



## Solving for the stationary distribution — Fisher Wright

- Equation becomes

$$(\delta'(1-y) - \delta y)\pi(y))' = \frac{1}{2} (y(1-y)\pi(y))''$$

- So for some constant  $c$

$$(\delta'(1-y) - \delta y)\pi(y) = \frac{1}{2} (y(1-y)\pi(y))' + c$$

- Now argue that at  $y = 0, 1$  left hand side should vanish; not trivial.
- Or just try to find a solution with  $\pi(0) = \pi(1) = 0$  so  $c = 0$ .
- This simplifies to

$$(\delta'(1-y) - \delta y)\pi(y) - (1-2y)\pi(y)/2 = y(1-y)\pi'(y)/2.$$



## Applied to Fisher Wright

- Divide by  $\pi(y)$  to get

$$\frac{(2\delta' - 1)(1 - y) - (2\delta - 1)y}{y(1 - y)} = \frac{\pi'(y)}{\pi(y)}.$$

- Integrate to get

$$\log \pi(y) = (2\delta' - 1) \ln(y) + (2\delta - 1) \ln(1 - y) + c$$

or

$$\pi(y) = Cy^{2\delta'-1}(1 - y)^{2\delta-1}$$

- This is a Beta( $2\delta'$ ,  $2\delta$ ) density giving  $C$  in terms of Gamma functions.



# Ito Calculus

- For ordinary calculus: if  $x(t)$  is a smooth (differentiable) function of  $t$  and  $f(x, t)$  is continuously differentiable in both arguments then

$$\begin{aligned}df(x(t), t) &= f_x(x(t), t)dx(t) + f_t(x(t), t)dt \\ &= (f_x(x(t), t)x'(t) + f_t(x(t), t)) dt\end{aligned}$$

by the ordinary rules of calculus.

- If  $x(t)$  is replaced by Brownian motion, however, then a change of  $\delta t$  in  $t$  changes  $f(x(t), t)$  by an amount proportional to  $\sqrt{\delta t}f_x(x(t), t)$  (with a random coefficient).
- And the next term in the Taylor expansion with respect to  $x$  is proportional to  $\delta t$ . Not negligible.
- The idea is

$$f(X(t) + dX_t, t) = f(X(t), t) + f_x(X(t), t)dX_t + \frac{1}{2}f_{xx}(X(t), t)(dX_t)^2$$

- And the  $(dX_t)^2$  term is like  $dt$ .



# Ito's Formula

- There are various versions of this formula. First version:
- $B_t$  is standard Brownian motion.
- $f(x, t)$  is twice differentiable in  $x$  and once in  $t$ .
- Then

$$df(B_t, t) = (f_t(B_t, t) + \frac{1}{2}f_{xx}(B_t, t))dt + f_x(B_t, t)dB_t$$

- This means

$$f(B_T, T) = f(0, 0) + \int_0^T \left( f_t(B_t, t) + \frac{1}{2}f_{xx}(B_t, t) \right) dt + \int_0^T f_x(B_t, t)dB_t.$$



## Example

- Take

$$f(x, t) = x_0 \exp\{(\alpha - \beta^2/2)t + \beta x\}$$

- Then

$$f_x(x, t) = \beta f(x, t)$$

$$f_{xx}(x, t) = \beta^2 f(x, t)$$

$$f_t(x, t) = (\alpha - \beta^2/2)f(x, t)$$

- So if  $X_t = f(B_t, t)$  we have

$$dX_t = df(B_t, t) = \left( (\alpha - \beta^2/2)X_t + \frac{1}{2}\beta^2 X_t \right) dt + \beta X_t dB_t.$$

- So Geometric Brownian motion solves

$$dX_t = \alpha X_t dt + \beta X_t dB_t$$



# Ito's Formula Generalized

- Second version:
- $B_t$  is standard Brownian motion.
- We have a process  $X$  which solves

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t$$

- $f(x, t)$  is twice differentiable in  $x$  and once in  $t$ .
- Then

$$df(X_t, t) = (f_t(X_t, t) + \mu(X_t, t)f_x(X_t, t) + \frac{\sigma^2(X_t, t)}{2}f_{xx}(X_t, t))dt + \sigma(X_t, t)f_x(X_t, t)dB_t$$

- Essentially we get  $f_x(X_t, t)dX_t$  and  $f_{xx}(X_t, t)(dX_t)^2/2$ .
- Then we ignore terms like  $dtdX_t$  and so on in squaring out.
- But we use  $(dB_t)^2 = dt$ .



## Second Example

- For Ornstein Uhlenbeck put

$$U_t = \int_0^t e^{\alpha s} dB_s$$

so

$$dU_t = e^{\alpha t} dB_t$$

- Then define

$$V_t = e^{-\alpha t} \left\{ V_0 + \sigma \int_0^t e^{\alpha s} dB_s \right\}.$$

- Define

$$f(x, t) = e^{-\alpha t} \{ V_0 + \sigma x \}.$$

- So

$$V_t = f(U_t, t).$$





## Example Continued

- We find

$$f_t(x, t) = -\alpha f(x, t)$$

$$f_x(x, t) = \sigma e^{\alpha t}$$

$$f_{xx}(x, t) = 0$$

- And that gives

$$dV_t = -\alpha V_t dt + \sigma dB_t.$$

- This is the Ornstein Uhlenbeck SDE as advertised.

