STAT 870

Problems: Assignment 2

1. Each day a random number of newspapers arrive at my house. The probability that k papers arrive is $p_k, k = 0, 1, \ldots$ At the end of the day I may or may not decide to throw out some of the papers that have accumulated. Given that there are m papers the probability that I throw j of them out is $1/(m+1)$ for $j = 0, \ldots, m$. Let X_n be the number of papers left at the end of day n (after I do the throwing out).

   (a) What must you assume in order to have X_n be a Markov chain?

   (b) Let μ be the expected number of papers delivered to my house on a given day. Show that
   $$\mathrm{E}(X_{n+1}|X_n) = \frac{X_n + \mu}{2}$$

   (c) Let ν be the long run expected number of papers I have at the end of a day. That is
   $$\nu = \lim_{n\to\infty} \mathrm{E}(X_n)$$
   Show that $\nu = \mu$.

2. Consider two urns A and B containing a total of N balls. An experiment is performed in which a ball is selected at random (all selections equally likely) at time $t(t = 1, 2, \ldots)$ from the collection of all N balls. Then an urn is selected at random (A is chosen with probability p and B is chosen with probability $q = 1-p$) and the ball previously drawn is placed in this urn. The state of the system at each trial is represented by the number of balls in A.

   (a) Determine the transition matrix for this Markov chain.

   (b) With the same setup suppose that an urn is chosen with probability proportional to the number of balls in the urn at time t. One of the balls in that urn is chosen at random and moved to an urn chosen at random with the probability that the destination urn will be urn A being k/N where k is the number of balls in A (this number is counted before the removing the selected ball from its urn). Determine the transition matrix for this Markov chain.

3. A coin is tossed until 2 successive heads appear. Find the expected number of tosses required by proceeding as follows: define a Markov chain $X_n; n = 0, 1, \ldots$ which records the state of the 2 most recent tosses. The state space is $\{HH, HT, TH, TT\}$. The first two tosses of the coin define X_0.

   (a) Write out the transition matrix for this chain if the probability of heads is p.

   (b) For each i and j in the state space compute μ_{ij} the mean time to get from state i to state j.

(c) By conditioning on X_0 relate the expected time to the first occurrence of a given pattern to the various μ_{ij}.

(d) Put all this together to compute the expected time to each of the four states.

4. A fair coin is tossed repeatedly. You are waiting for the pattern THHH and I am waiting for the pattern HHHH. What is the chance you are finished waiting before I am?

5. Customers arrive at a facility and wait there until a total of K customers have accumulated. Upon the arrival of the Kth customer all customers are served instantaneously and the process repeats. Let $\xi_0, \xi_1, \ldots$ be the number of customers arriving in successive periods and assume that the ξ are independent Bernoulli(α) random variables (that is, in each time period either 1 or 0 customers arrives). Let X_n be the number of customers waiting at time n. Then $\{X_n\}$ is a Markov chain with states $0, 1, \ldots, K-1$. Write out the transition matrix when $K = 3$.

6. At the end of a month a large retail store classifies each receivable account in one of four classes: current, 30 to 60 days overdue, 60 to 90 days overdue or over 90 days overdue. Assume that an account is paid in full with probability 0.95, 0.50, 0.20 and 0.10 depending on its status last month. In the long run what fraction of accounts are over 90 days overdue?

7. Suppose $X_m; 0 \le m \le n$ is a Markov chain with stationary transition matrix $\mathbf{P}$ and initial distribution α.

   (a) Compute
   $$P(X_{k-1} = j | X_k = i, X_{k+1} = i_{k+1}, \ldots, X_n = i_n)$$
   in terms of $\mathbf{P}$ and α.

   (b) Set $Y_k = X_{n-k}; k = 0, \ldots, n$. Show that Y is a Markov Chain (the transitions are not stationary) and describe the initial distribution of Y.

   (c) Show that if $\alpha = \pi$, the stationary initial distribution of X, then Y has stationary transitions and express the transition matrix $\mathbf{Q}$ of Y in terms of $\mathbf{P}$ and π.

   (d) **Note**: if $\mathbf{P} = \mathbf{Q}$ then the process is called **time reversible**.

8. Suppose $X_n; n \ge 0$ is a two state Markov chain with transition matrix
   $$\mathbf{P} = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$$
   Define $Y_n = (X_n, X_{n+1})$ and show that Y_n is a Markov chain. What is the transition matrix of Y?

9. Suppose π is some probability vector on a state space S and $\mathbf{P}$ is a transition matrix on S. Let X_n be the following Markov Chain:

- X_0 has some initial distribution α.
- If $X_n = i$ then let J be a random variable with

$$P(J = j | X_n = i) = \mathbf{P}_{ij}$$

- Let U be a Uniform[0,1] random variable and put

$$X_{n+1} = \begin{cases} J & U \le \min\{\pi_j \mathbf{P}_{ji} / \pi_i \mathbf{P}_{ij}, 1\} \\ i & \text{otherwise} \end{cases}$$

Prove that X_n is a Markov chain with stationary initial distribution π. As you do this please describe the transition matrix of X_n.

10. Suppose $\epsilon_1, \epsilon_2, \ldots$ is a sequence of iid standard normal variables. Let X_0 have a $N(0, \tau^2)$ distribution for some τ and define

$$X_{t+1} = \rho X_t + \epsilon_{t+1}$$

for $t = 0, 1, \cdots$.

(a) This defines a continuous state space Markov chain; explain why by finding the conditional density of X_{t+1} given $X_0, \ldots, X_t$.

(b) For some values of ρ there is a choice of τ which provides a stationary initial distribution for this chain. Compare the variances of X_0 and X_1 to find the stationary initial distribution for this chain and to discover for which values of ρ this stationary initial distribution exists.

11. Suppose X and Y are random variables with mean 0 and finite variances. Suppose $E(Y|X) = X$ and $E(X|Y) = Y$. Deduce that $P(X = Y) = 1$.

12. **Random walk on a graph.** Consider a regular polygon with $n+1$ vertices numbered from 0 to n. Start from 0 at time 0. At each time point move clockwise with probability p and counterclockwise with probability $1-p$. Stop the first time you get back to vertex 0. What is the probability that you visit all the other vertices before you stop?

13. You toss a fair die repeatedly getting results X_1, X_2 and so on. Let $Y_n = X_1 + \cdots + X_n$. Fix an integer $k > 1$. Compute

$$\lim_{n\to\infty} P(Y_n \text{ is divisible by } k).$$

Hint: Markov chain. It might help to think about

$$\lim_{n\to\infty} P(Y_n \text{ leaves remainder } r \text{ when divided by } k).$$

14. **Bienaymé-Galton-Watson processes** I am going to describe this process in terms of fathers and sons but there are application in epidemics, in describing chain reactions and in many other places. Suppose you start with 1 man at generation 0. Generation 1 consists of his sons. Let p_j be the probability he has j sons. Generation 2 consists of sons of sons and we suppose each generation 1 son has probability p_j of having j sons independently of what happens to other sons and anything happening at an earlier generation. In general, generation $n+1$ consists of all the sons of those in generation n with all sons behaving independently with identical distributions for the number of (male) offspring. Let X_n be the size of generation n.

    (a) The probability generating function of a random variable Y taking values in $\{0, 1, \cdots\}$ is

    $$E(s^Y) = \sum_{j=0}^{\infty} s^j P(Y = j)$$

    Let $\phi(s)$ be the probability generating function of X_1, that is,

    $$\phi(s) = E(s^{X_1}) = \sum_{j=0}^{\infty} s^j p_j.$$

    Find the probability generating function, ϕ_n, of X_n in terms of ϕ by proving $\phi_n(s) = \phi(\phi_{n-1}(s)) = \phi_{n-1}(\phi(s))$.

    (b) Give a simple formula for $q_n \equiv P(X_n = 0)$ in terms of ϕ_n.

    (c) Show that $q_1 < q_2 < \cdots$ and then argue that q_n has a limit

    $$q = P(\text{there exists } n \text{ such that } X_n = 0).$$

    (d) Use the preceding three parts to find an equation involving ϕ and q which might be solved to find q if you knew ϕ.

    (e) Find q in the special case where the number of sons has a Binomial distribution with $n = 2$ and some success probability p.

15. Consider the following strategy for comparing two medical treatments, say treatment A and treatment B. Patients are treated one at a time and the result of each treatment is recorded as a Success or a Failure. Every time a treatment succeeds the next patient is treated with the same treatment which was just successful. When a treatment fails, the next patient is treated with the other treatment. Suppose that the probability that treatment A succeeds is p_A while the probability that treatment B succeeds is p_B. In the long run what fraction of patients are treated with treatment B?

16. With the same set up as in the previous question suppose that the treatment is changed only after two consecutive failures. Using the four states:

    0 About to use A, last trial was not a failure with A.

    1 About to use A, last trial was a failure with A.

2 About to use B, last trial was not a failure with B.

3 About to use B, last trial was a failure with B.

give the transition matrix of a Markov Chain which can be used to determine what fraction of patients are treated with treatment B in the long run and show clearly what equations you would solve to find the answer. I don't want the equations left in matrix form.

17. Which of the designs in the previous two questions is better and why?

18. A commuter has two possible routes to work, A and B. There is construction activity on route A about 1 day in 20, and on route B about 1 day in 10. If the commuter takes route A and finds construction she switches to route B for the next day otherwise she uses A again. If the commuter takes route B and finds construction she switches to route A for the next day; otherwise she uses B again.

    In the long run on what fraction of days does she commute via route A and on what fraction of her trips does she find construction?

Due: mid June 2013.