STAT 870

Problems: Assignment 3

1. A container contains n particles; some are black and the rest are white. Assume that collisions between particles in the container occur at the times of a Poisson process with rate λ. When a collision occurs it is equally likely to be between any of the $n(n-1)/2$ pairs of particles. When two black particles collide there is a chance p that (precisely) one will turn white (instantaneously); otherwise they both stay black. When two white particles collide there is a chance q that (precisely) one will turn black (instantaneously); otherwise they both stay white. When two opposite colour particles collide there is a chance $r/2$ that they both become white and a chance $r/2$ that they both become black.

   (a) Describe an appropriate Markov Chain to analyze this system. Give the transition matrix for the skeleton chain, the instantaneous transition rates and the mean holding times in each state.

   (b) For $n = 3$ what fraction of the time are all three balls the same colour?

2. Customers arrive at a facility according to a Poisson Process with rate λ. There is a waiting time cost of c per unit time for each customer waiting. At fixed times T, $2T$, $3T$, and so on the customers are "processed" (so that the customer's waiting ends). The cost of dispatch is K.

   (a) What is the expected cost of the first cycle from time 0 to time T counting both dispatch cost and customer waiting cost.

   (b) What value of T minimizes the expected cost per unit time?

3. Assume that points are scattered in the plane according to a Poisson counting process with rate λ. Around each point we draw a circle with random radius R_i independently of the location in the plane of the centre. Assume that R has density f and finite second moment τ^2.

   (a) If $C(r)$ is the number of circles which cover the origin in the plane and have centres located at a distance less than r from the origin show that C is an inhomogeneous Poisson process with intensity $\lambda_r = 2\pi\lambda r \int_r^\infty f(u)\,du$.

   (b) Show that $C(\infty)$, which is the number of circles covering the origin, has a Poisson distribution with parameter $\lambda\pi\tau^2$.

4. **Minimal Spanning Trees**. You have n nodes to be connected together. You can connect node i to node j for a cost of C_{ij}. We model the C_{ij} as having independent exponential distributions with mean 1; assume $C_{ij} = C_{ji}$. We use the following algorithm: Find the pair which is cheapest to connect and connect them. Then find the cheapest connection we can make between a node which is not yet connected and a node which is connected. Repeat for a total of $n-1$ times. At this point the nodes are all connected. What is the expected costs of these connections for $n = 3$ and $n = 4$?

5. You are standing on a corner watching cars come along the street waiting to cross. Cars arrive according to a Poisson process. You cross the street when you can see that there will not be a car crossing your path in the next T time units. Find, in terms of T and the Poisson arrival rate:
   (a) The probability you won't have to wait at all.
   (b) The expected waiting time.

6. You want to take the last bus home before midnight. Sadly, buses arrive according to a Poisson process with rate λ. It is now T minutes before midnight. You adopt the following strategy: wait until s minutes before midnight and then catch the next bus after that. You lose if there is no bus between then and midnight or if the bus you take is not the last one before midnight.
   (a) What is the probability of winning?
   (b) What value of s maximizes this probability?
   (c) What is the maximized probability of winning?

7. Suppose we model the number of bugs in a piece of software as having a Poisson distribution with mean λ. At time 0 the software is put into use. Suppose that each bug is detected after a random amount of time. These times are independent and identically distributed with cumulative distribution G. Bugs are fixed instantly upon detection.

   What is the joint distribution of the number of detected and undetected bugs at time t?

8. Buses arrive according to a Poisson process with rate λ. I arrive at the stop at noon. Given that the second bus to arrive after noon comes between 1 and 2 PM what is the chance that the first bus to arrive after noon comes before 1?

9. Suppose that SFU graduates 3 PhDs in statistics per year, on average. These PhDs then have careers which last an average of 35 years. How many SFU PhD statistics graduates should be active in their careers in the year 2100?

10. In a similar vein the population of BC is about 4 million. How many children are in Grade 12 this year? In this question and the previous I hope to see a short discussion of the issues – just a list of obvious points will do – but I also hope to see an estimate and an explanation. No need to do any research to discover the correct answer.

11. Here is a simplistic model for the spread of an infection in a population of n people. Person i contacts Person j at rate λ. This happens independently for all possible pairs of persons. If one of two persons coming into contact is infected but the other is not the uninfected person becomes infected with probability p. Once infected a person stays that way forever.

(a) Describe the number of infected people as a Markov Chain giving all the relevant rates.

(b) Describe the skeleton chain.

(c) Compute the expected time till everyone is infected starting from 1 individual.

(d) Now suppose there are two types of individuals, say N_i of type i for $i = 1, 2$. For individuals of type i coming into contact with an infected individual the probability of becoming infected is p_i. Find a new Markov chain to compute the expected waiting time till everyone is infected. I do not expect you to be able to finish the computation in general – just set up equations to solve.

12. In the lecture notes I described a model for nuclear fission. You start with a large number atoms of some radioactive element and a single neutron. After an exponentially distributed amount of time with rate λ the neutron either collides with a nucleus of the radioactive element or exits the sample. The probability of collision is p. Once a collision happens the neutron is absorbed into the nucleus which then splits into 2 lighter nuclei and 2 neutrons (so the one neutron has been replaced by 2). Each neutron repeats the process independently of all other neutrons. Model the number of neutrons in the sample at time t as a Markov chain describing all the relevant rates. Compute the mean number of neutrons present at time t and find the limit as $t \to \infty$ of the number of neutrons.

13. In the last problem I had you ignore the fact that each fission uses up one of the initial nuclei. Write down a relevant Markov chain model for the pair (number of neutrons, number of heavy nuclei) if the probability of collision is p_n when there are n heavy nuclei and the probability of exiting the sample is $1 - p_n$.

14. Going back to the previous question, once again ignore the fact you are using up the heavy nuclei. Now however suppose that when the nucleus splits into two pieces the nature of the pieces is random and that the number of neutrons emitted is 2 with probability p_2 and 3 with probability p_3 and $p_2 + p_3 = 1$. Once again describe the relevant Markov chain.

15. Old telephone exchanges could handle a fixed number, say K, of calls. Calls arrive according to a Poisson process with rate λ and last for an exponential amount of time with rate μ. When the exchange is handling K calls any arriving call is lost to the system. What is the stationary distribution of the number of calls being handled and what is the long run fraction of calls lost?

16. Two systems are maintained by one tech person. The systems each go down at at rate μ independently of each other. When one breaks the tech person takes an exponentially distributed amount of time to fix it; the rate is λ. If the other system goes down while the first one is down it waits to start service until the first one is fixed. What is the stationary distribution of the number of working systems?

17. If the machines had rates μ_1 and μ_2 and the repair person had a first in first out service policy how would you compute the long run fraction of the time that both machines are out of service?

Due: early July 2013