

# Brownian Motion

Richard Lockhart

Simon Fraser University

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# Purposes of Today's Lecture

- Describe Brownian motion as a limit of random walks.
- Define Brownian motion.
- Describe properties of Brownian motion.
- Use reflection principle to deduce law of maximum.
- Define martingales.
- Derive Black-Scholes formula.



# Brownian Motion

- For fair random walk  $Y_n =$  number of heads minus number of tails,

$$Y_n = U_1 + \cdots + U_n$$

where the  $U_i$  are independent and

$$P(U_i = 1) = P(U_i = -1) = \frac{1}{2}$$

- Notice:

$$E(U_i) = 0$$

$$\text{Var}(U_i) = 1$$

- Recall central limit theorem:

$$\frac{U_1 + \cdots + U_n}{\sqrt{n}} \Rightarrow N(0, 1)$$

- Now: rescale time axis so that  $n$  steps take 1 time unit and vertical axis so step size is  $1/\sqrt{n}$ .



# Brownian Motion Graph

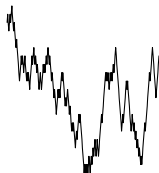
$n=16$



$n=64$



$n=256$



$n=1024$



## Limit of Random Walks

- We now turn these pictures into a stochastic process:
- For  $\frac{k}{n} \leq t < \frac{k+1}{n}$  we define

$$X_n(t) = \frac{U_1 + \cdots + U_k}{\sqrt{n}}$$

- Notice:

$$E(X_n(t)) = 0$$

and

$$\text{Var}(X_n(t)) = \frac{k}{n}$$

- As  $n \rightarrow \infty$  with  $t$  fixed we see  $k/n \rightarrow t$ . Moreover:

$$\frac{U_1 + \cdots + U_k}{\sqrt{k}} = \sqrt{\frac{n}{k}} X_n(t)$$

converges to  $N(0, 1)$  by the central limit theorem. Thus

$$X_n(t) \Rightarrow N(0, t)$$



# Limit of Random Walks

- Also:  $X_n(t + s) - X_n(t)$  is independent of  $X_n(t)$  because the 2 rvs involve sums of different  $U_j$ .
- Conclusions: As  $n \rightarrow \infty$  the processes  $X_n$  converge to a process  $X$  with the properties:
  - 1  $X(t)$  has a  $N(0, t)$  distribution.
  - 2  $X$  has independent increments: if

$$0 = t_0 < t_1 < t_2 < \dots < t_k$$

then

$$X(t_1) - X(t_0), \dots, X(t_k) - X(t_{k-1})$$

are independent.

- 3 The increments are **stationary**: for all  $s$

$$X(t + s) - X(s) \sim N(0, t)$$

- 4  $X(0) = 0$ .



## Definition of Brownian Motion

**Def'n:** Any process satisfying 1-4 above is a Brownian motion.

### Properties of Brownian motion

- Suppose  $t > s$ . Then

$$\begin{aligned}E(X(t)|X(s)) &= E\{X(t) - X(s) + X(s)|X(s)\} \\ &= E\{X(t) - X(s)|X(s)\} + E\{X(s)|X(s)\} \\ &= 0 + X(s) = X(s)\end{aligned}$$

Notice the use of independent increments and of  $E(Y|Y) = Y$ .

- Again if  $t > s$ :

$$\begin{aligned}\text{Var}\{X(t)|X(s)\} &= \text{Var}\{X(t) - X(s) + X(s)|X(s)\} \\ &= \text{Var}\{X(t) - X(s)|X(s)\} \\ &= \text{Var}\{X(t) - X(s)\} \\ &= t - s\end{aligned}$$



# Conditional Distributions

- Suppose  $t < s$ . Then  $X(s) = X(t) + \{X(s) - X(t)\}$  is a sum of two independent normal variables. Do following calculation:
- $X \sim N(0, \sigma^2)$ , and  $Y \sim N(0, \tau^2)$  independent.  $Z = X + Y$ .
- Compute conditional distribution of  $X$  given  $Z$ :

$$\begin{aligned} f_{X|Z}(x|z) &= \frac{f_{X,Z}(x,z)}{f_Z(z)} \\ &= \frac{f_{X,Y}(x, z-x)}{f_Z(z)} \\ &= \frac{f_X(x)f_Y(z-x)}{f_Z(z)} \end{aligned}$$





# Conditional Distributions

- Now  $Z$  is  $N(0, \gamma^2)$  where  $\gamma^2 = \sigma^2 + \tau^2$  so

$$\begin{aligned} f_{X|Z}(x|z) &= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)} \frac{1}{\tau\sqrt{2\pi}} e^{-(z-x)^2/(2\tau^2)}}{\frac{1}{\gamma\sqrt{2\pi}} e^{-z^2/(2\gamma^2)}} \\ &= \frac{\gamma}{\tau\sigma\sqrt{2\pi}} \exp\{-(x-a)^2/(2b^2)\} \end{aligned}$$

for suitable choices of  $a$  and  $b$ . To find them compare coefficients of  $x^2$ ,  $x$  and  $1$ .



# Conditional Distributions

- Coefficient of  $x^2$ :

$$\frac{1}{b^2} = \frac{1}{\sigma^2} + \frac{1}{\tau^2}$$

so  $b = \tau\sigma/\gamma$ .

- Coefficient of  $x$ :

$$\frac{a}{b^2} = \frac{z}{\tau^2}$$

so that

$$a = b^2 z / \tau^2 = \frac{\sigma^2}{\sigma^2 + \tau^2} z$$

- Finally you should check that

$$\frac{a^2}{b^2} = \frac{z^2}{\tau^2} - \frac{z^2}{\gamma^2}$$

to make sure the coefficients of 1 work out as well.

- So given  $Z = z$  conditional distribution of  $X$  is  $N(a, b^2)$ .



## Application to Brownian motion

- For  $t < s$  let  $X$  be  $X(t)$  and  $Y$  be  $X(s) - X(t)$  so  $Z = X + Y = X(s)$ .
- Then  $\sigma^2 = t$ ,  $\tau^2 = s - t$  and  $\gamma^2 = s$ .
- Thus

$$b^2 = \frac{(s-t)t}{s}$$

and

$$a = \frac{t}{s}X(s)$$

- So:

$$E(X(t)|X(s)) = \frac{t}{s}X(s)$$

and

$$\text{Var}(X(t)|X(s)) = \frac{(s-t)t}{s}$$



# The Reflection Principle

- Tossing a fair coin:  
HTHHHTHTHHHTHHHTTHTH      5 more heads than tails
- THTTTHTHTTTHTTTHTHT      5 more tails than heads
- Both sequences have the same probability.
- So: for random walk starting at stopping time:
- Any sequence with  $k$  more heads than tails in next  $m$  tosses is matched to sequence with  $k$  more tails than heads. Both sequences have same prob.
- Suppose  $Y_n$  is a fair ( $p = 1/2$ ) random walk. Define

$$M_n = \max\{Y_k, 0 \leq k \leq n\}$$



## Compute $P(M_n \geq x)$ ?

- Trick: Compute

$$P(M_n \geq x, Y_n = y)$$

- First: if  $y \geq x$  then

$$\{M_n \geq x, Y_n = y\} = \{Y_n = y\}$$

- Second: if  $M_n \geq x$  then

$$T \equiv \min\{k : Y_k = x\} \leq n$$

- Fix  $y < x$ . Consider a sequence of H's and T's which leads to say  $T = k$  and  $Y_n = y$ .
- Switch the results of tosses  $k + 1$  to  $n$  to get a sequence of H's and T's which has  $T = k$  and  $Y_n = x + (x - y) = 2x - y > x$ . This proves

$$P(T = k, Y_n = y) = P(T = k, Y_n = 2x - y)$$



## Computation Continued

- This is true for each  $k$  so

$$\begin{aligned}P(M_n \geq x, Y_n = y) &= P(M_n \geq x, Y_n = 2x - y) \\ &= P(Y_n = 2x - y)\end{aligned}$$

- Finally, sum over all  $y$  to get

$$P(M_n \geq x) = \sum_{y \geq x} P(Y_n = y) + \sum_{y < x} P(Y_n = 2x - y)$$

- Make the substitution  $k = 2x - y$  in the second sum to get

$$\begin{aligned}P(M_n \geq x) &= \sum_{y \geq x} P(Y_n = y) + \sum_{k > x} P(Y_n = k) \\ &= 2 \sum_{k > x} P(Y_n = k) + P(Y_n = x)\end{aligned}$$



## Brownian motion version

- The supremum and hitting time for level  $x$  are:

$$M_t = \max\{X(s); 0 \leq s \leq t\}$$

$$T_x = \min\{s : X(s) = x\}$$

- Then

$$\{T_x \leq t\} = \{M_t \geq x\}$$

- Any path with  $T_x = s < t$  and  $X(t) = y < x$  is matched to an equally likely path with  $T_x = s < t$  and  $X(t) = 2x - y > x$ .
- So for  $y > x$

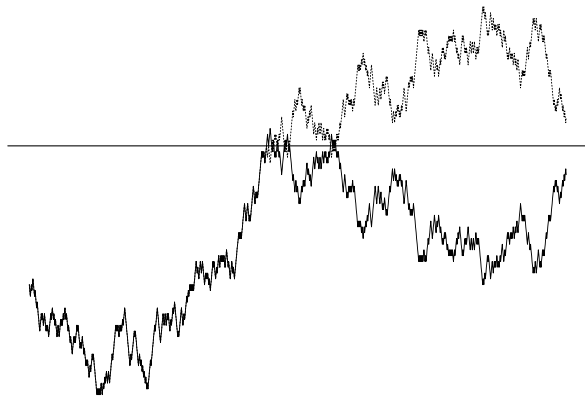
$$P(M_t \geq x, X(t) > y) = P(X(t) > y)$$

while for  $y < x$

$$P(M_t \geq x, X(t) < y) = P(X(t) > 2x - y)$$



# Reflection Principal Graphically





## Strong Markov Property

- A random variable  $T$  which is non-negative (or possibly  $+\infty$ ) is a stopping time for Brownian motion if

$$\{T \leq t\} \in \mathcal{H}_t = \sigma\{B(u); 0 \leq u \leq t\}.$$

- The first time  $T_x$  that  $B_t = x$  is a stopping time.
- For any stopping time  $T$  the process

$$t \mapsto B(T + t) - B(T)$$

is a Brownian motion.

- The future of the process from  $T$  on is like the process started at  $B(T)$  at  $t = 0$ .
- Brownian motion is symmetric: if  $B$  is a Brownian motion so is  $-B$ .
- So

$$W(t) = \begin{cases} B_t & t < T \\ B(T) - (B(T + t) - B(T)) & t \geq T \end{cases}$$

is a Brownian motion.

- This proves the reflection principle.



## Reflection Principle Continued

- Let  $y \rightarrow x$  to get

$$\begin{aligned}P(M_t \geq x, X(t) > x) &= P(M_t \geq x, X(t) < x) \\ &= P(X(t) > x)\end{aligned}$$

- Adding these together gives

$$\begin{aligned}P(M_t > x) &= 2P(X(t) > x) \\ &= 2P(N(0, 1) > x/\sqrt{t})\end{aligned}$$

- Hence  $M_t$  has the distribution of  $|N(0, t)|$ .



## Reflection

- On the other hand in view of

$$\{T_x \leq t\} = \{M_t \geq x\}$$

the density of  $T_x$  is

$$\frac{d}{dt} 2P(N(0,1) > x/\sqrt{t})$$

- Use the chain rule to compute this.
- First

$$\frac{d}{dy} P(N(0,1) > y) = -\phi(y)$$

where  $\phi$  is the standard normal density

$$\phi(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

because  $P(N(0,1) > y)$  is 1 minus the standard normal cdf.



# First Passage Time Law

- So

$$\begin{aligned}\frac{d}{dt}2P(N(0,1) > x/\sqrt{t}) \\ &= -2\phi(x/\sqrt{t})\frac{d}{dt}(x/\sqrt{t}) \\ &= \frac{x}{\sqrt{2\pi}t^{3/2}}\exp\{-x^2/(2t)\}\end{aligned}$$

- This density is called the **Inverse Gaussian** density.
- $T_x$  is called a **first passage time**
- NOTE: the preceding is a density when viewed as a function of the variable  $t$ .



# Martingales

**Def'n:** A stochastic process  $M(t)$  indexed by either a discrete or continuous time parameter  $t$  is a **martingale** if:

$$E\{M(t)|M(u); 0 \leq u \leq s\} = M(s)$$

whenever  $s < t$ .



# Examples of Martingales

- A fair random walk is a martingale.
- If  $N(t)$  is a Poisson Process with rate  $\lambda$  then  $N(t) - \lambda t$  is a martingale.
- Standard Brownian motion (defined above) is a martingale.
- Brownian motion with drift is a process of the form

$$X(t) = \sigma B(t) + \mu t$$

where  $B$  is **standard** Brownian motion, introduced earlier.

- $X$  is a martingale if  $\mu = 0$ .
- We call  $\mu$  the **drift**.



## More Examples

- If  $X(t)$  is a Brownian motion with drift then

$$Y(t) = e^{X(t)}$$

is a geometric Brownian motion.

- For suitable  $\mu$  and  $\sigma$  we can make  $Y(t)$  a martingale.
- If a gambler makes a sequence of fair bets and  $M_n$  is the amount of money s/he has after  $n$  bets then  $M_n$  is a martingale – even if the bets made depend on the outcomes of previous bets, that is, even if the gambler plays a strategy.



## Some evidence for some of the above

- Random walk:  $U_1, U_2, \dots$  iid with

$$P(U_i = 1) = P(U_i = -1) = 1/2$$

and  $Y_k = U_1 + \dots + U_k$  with  $Y_0 = 0$ .

- Then

$$\begin{aligned} E(Y_n | Y_0, \dots, Y_k) &= E(Y_n - Y_k + Y_k | Y_0, \dots, Y_k) \\ &= E(Y_n - Y_k | Y_0, \dots, Y_k) + Y_k \\ &= \sum_{k+1}^n E(U_j | U_1, \dots, U_k) + Y_k \\ &= \sum_{k+1}^n E(U_j) + Y_k \\ &= Y_k \end{aligned}$$





## Things to notice

- $Y_k$  treated as constant given  $Y_1, \dots, Y_k$ .
- Knowing  $Y_1, \dots, Y_k$  is equivalent to knowing  $U_1, \dots, U_k$ .
- For  $j > k$  we have  $U_j$  independent of  $U_1, \dots, U_k$  so conditional expectation is unconditional expectation.
- Since Standard Brownian Motion is limit of such random walks we get martingale property for standard Brownian motion.



## Another martingale

**Poisson Process:**  $X(t) = N(t) - \lambda t$ . Fix  $t > s$ .

$$\begin{aligned} E(X(t)|X(u); 0 \leq u \leq s) &= E(X(t) - X(s) + X(s)|\mathcal{H}_s) \\ &= E(X(t) - X(s)|\mathcal{H}_s) + X(s) \\ &= E(N(t) - N(s) - \lambda(t - s)|\mathcal{H}_s) + X(s) \\ &= E(N(t) - N(s)) - \lambda(t - s) + X(s) \\ &= \lambda(t - s) - \lambda(t - s) + X(s) \\ &= X(s) \end{aligned}$$

Things to notice:

- I used independent increments.
- $\mathcal{H}_s$  is shorthand for the conditioning event.
- Similar to random walk calculation.



# Black Scholes

- We model the price of a stock as

$$X(t) = x_0 e^{Y(t)}$$

where

$$Y(t) = \sigma B(t) + \mu t$$

is a Brownian motion with drift ( $B$  is standard Brownian motion).

- If annual interest rates are  $e^\alpha - 1$  we call  $\alpha$  the instantaneous interest rate; if we invest \$1 at time 0 then at time  $t$  we would have  $e^{\alpha t}$ .
- In this sense an amount of money  $x(t)$  to be paid at time  $t$  is worth only  $e^{-\alpha t} x(t)$  at time 0 (because that much money at time 0 will grow to  $x(t)$  by time  $t$ ).



## Present Value

- If the stock price at time  $t$  is  $X(t)$  per share then the present value of 1 share to be delivered at time  $t$  is

$$Z(t) = e^{-\alpha t} X(t)$$

- With  $X$  as above we see

$$Z(t) = x_0 e^{\sigma B(t) + (\mu - \alpha)t}$$

- Now we compute

$$\mathbb{E}\{Z(t)|Z(u); 0 \leq u \leq s\} = \mathbb{E}\{Z(t)|B(u); 0 \leq u \leq s\}$$

for  $s < t$ .

- Write

$$Z(t) = x_0 e^{\sigma B(s) + (\mu - \alpha)t} \times e^{\sigma(B(t) - B(s))}$$

- Since  $B$  has independent increments we find

$$\mathbb{E}\{Z(t)|B(u); 0 \leq u \leq s\} = x_0 e^{\sigma B(s) + (\mu - \alpha)t} \mathbb{E}\left[e^{\sigma\{B(t) - B(s)\}}\right]$$



# Moment Generating Functions

- Note:  $B(t) - B(s)$  is  $N(0, t - s)$ ; the expected value needed is the moment generating function of this variable at  $\sigma$ .
- Suppose  $U \sim N(0, 1)$ . The Moment Generating Function of  $U$  is

$$M_U(r) = E(e^{rU}) = e^{r^2/2}$$

- Rewrite

$$\sigma\{B(t) - B(s)\} = \sigma\sqrt{t - s}U$$

where  $U \sim N(0, 1)$  to see

$$E\left[e^{\sigma\{B(t)-B(s)\}}\right] = e^{\sigma^2(t-s)/2}$$

- Finally we get

$$\begin{aligned} E\{Z(t)|Z(u); 0 \leq u \leq s\} &= x_0 e^{\sigma B(s) + (\mu - \alpha)s} e^{(\mu - \alpha)(t-s) + \sigma^2(t-s)/2} \\ &= Z(s) \end{aligned}$$

provided

$$\mu + \sigma^2/2 = \alpha.$$



# Option Pricing

- If this identity is satisfied then the present value of the stock price is a martingale.
- Suppose you can pay  $\$c$  today for the right to pay  $K$  for a share of this stock at time  $t$  (regardless of the actual price at time  $t$ ).
- If, at time  $t$ ,  $X(t) > K$  you will **exercise** your **option** and buy the share making  $X(t) - K$  dollars.
- If  $X(t) \leq K$  you will not exercise your option; it becomes worthless.
- The present value of this option is

$$e^{-\alpha t}(X(t) - K)_+ - c$$

where

$$z_+ = \begin{cases} z & z > 0 \\ 0 & z \leq 0 \end{cases}$$

(Called **positive part** of  $z$ .)



## In a fair market

- The discounted share price  $e^{-\alpha t}X(t)$  is a martingale.
- The expected present value of the option is 0.
- So:

$$c = e^{-\alpha t} \mathbb{E} [\{X(t) - K\}_+]$$

- Since

$$X(t) = x_0 e^{N(\mu t, \sigma^2 t)}$$

we are to compute

$$\mathbb{E} \left\{ \left( x_0 e^{\sigma t^{1/2} U + \mu t} - K \right)_+ \right\}$$



## Black-Scholes Continued

- This is

$$\int_a^{\infty} \left( x_0 e^{bu+d} - K \right) e^{-u^2/2} du / \sqrt{2\pi}$$

where

$$a = (\log(K/x_0) - \mu t) / (\sigma t^{1/2}), b = \sigma t^{1/2}, d = \mu t$$

- Evidently

$$K \int_a^{\infty} e^{-u^2/2} du / \sqrt{2\pi} = KP(N(0, 1) > a)$$

- The other integral needed is

$$\begin{aligned} \int_a^{\infty} e^{-u^2/2+bu} du / \sqrt{2\pi} &= \int_a^{\infty} \frac{e^{-(u-b)^2/2} e^{b^2/2}}{\sqrt{2\pi}} du \\ &= \int_{a-b}^{\infty} \frac{e^{-v^2/2} e^{b^2/2}}{\sqrt{2\pi}} dv \\ &= e^{b^2/2} P(N(0, 1) > a - b) \end{aligned}$$





# Black-Scholes Continued

- Introduce the notation

$$\Phi(v) = P(N(0, 1) \leq v) = P(N(0, 1) > -v)$$

and do all the algebra to get

$$\begin{aligned}c &= \left\{ e^{-\alpha t} e^{b^2/2+d} x_0 \Phi(b-a) - Ke^{-\alpha t} \Phi(-a) \right\} \\ &= x_0 e^{(\mu+\sigma^2/2-\alpha)t} \Phi(b-a) - Ke^{-\alpha t} \Phi(-a) \\ &= x_0 \Phi(b-a) - Ke^{-\alpha t} \Phi(-a)\end{aligned}$$

- This is the Black-Scholes option pricing formula.

