Continuous Time Markov Chains

Richard Lockhart

Simon Fraser University

STAT 870 — Summer 2013

Purposes of Today's Lecture

- Define Continuous Time Markov Chain.
- Prove Chapman-Kolmogorov equations.
- Deduce Kolmogorov's forward and backward equations.
- Review properties.
- Define skeleton chain.
- Discuss ergodic theorem.

Continuous Time Markov Chains

- Consider a population of single celled organisms in a stable environment.
- Fix short time interval, length h.
- Each cell has some prob of dividing to produce 2, some other prob of dying.
- We might suppose:
  - Different organisms behave independently.
  - Probability of division (for specified organism) is λh plus $o(h)$.
  - Probability of death is μh plus $o(h)$.
  - Prob an organism divides twice (or divides once and dies) in interval of length h is $o(h)$.

Basic Notation

- Notice tacit assumption: constants of proportionality do not depend on time.
- That is our interpretation of "stable environment".
- Notice too that we have taken the constants not to depend on which organism we are talking about.
- We are really assuming that the organisms are all similar and live in similar environments.
- Notation: $Y(t)$: total population at time t.
- Notation: $\mathcal{H}_t$: history of the process up to time t.
- Notation: We generally take

$$\mathcal{H}_t = \sigma\{Y(s); 0 \le s \le t\}$$

Histories or Filtrations

Def'n: General definition of a **history** (alternative jargon **filtration**): any family of σ-fields indexed by t satisfying:

- $s < t$ implies $\mathcal{H}_s \subset \mathcal{H}_t$.
- $Y(t)$ is a $\mathcal{H}_t$ measurable random variable.
- $\mathcal{H}_t = \bigcap_{s>t} \mathcal{H}_s$.

The last assumption is a technical detail we will ignore from now on.

Modelling Assumptions

- Condition on event $Y(t) = n$.
- Then the probability of two or more divisions (either more than one division by a single organism or two or more organisms dividing) is $o(h)$ by our assumptions.
- Similarly the probability of both a division and a death or of two or more deaths is $o(h)$.
- So probability of exactly 1 division by any one of the n organisms is $n\lambda h + o(h)$.
- Similarly probability of 1 death is $n\mu h + o(h)$.

The Markov Property

- We deduce:

$$
\begin{aligned}
P(Y(t+h) &= n+1 | Y(t) = n, \mathcal{H}_t) \\
&= n\lambda h + o(h) \\
P(Y(t+h) &= n-1 | Y(t) = n, \mathcal{H}_t) \\
&= n\mu h + o(h) \\
P(Y(t+h) &= n | Y(t) = n, \mathcal{H}_t) \\
&= 1 - n(\lambda + \mu)h + o(h) \\
P(Y(t+h) &\notin \{n-1, n, n+1\} | Y(t) = n, \mathcal{H}_t) \\
&= o(h)
\end{aligned}
$$

- These equations lead to:

$$
\begin{aligned}
P(Y(t+s) = j | Y(s) = i, \mathcal{H}_s) &= P(Y(t+s) = j | Y(s) = i) \\
&= P(Y(t) = j | Y(0) = i)
\end{aligned}
$$

- This is the **Markov Property**.

Definitions

Def'n: A process $\{Y(t); t \ge 0\}$ taking values in S, a finite or countable state space is a Markov Chain if

$$\begin{aligned} P(Y(t+s) = j | Y(s) = i, \mathcal{H}_s) \\ &= P(Y(t+s) = j | Y(s) = i) \\ &\equiv \mathbf{P}_{ij}(s, s+t) \end{aligned}$$

Def'n: A Markov chain Y has **stationary transitions** if

$$\mathbf{P}_{ij}(s, s+t) = \mathbf{P}_{ij}(0, t) \equiv \mathbf{P}_{ij}(t)$$

From now on: our chains have stationary transitions.

Summary of Markov Process Results

- Chapman-Kolmogorov equations:

$$\mathbf{P}_{ik}(t+s) = \sum_j \mathbf{P}_{ij}(t)\mathbf{P}_{jk}(s)$$

- Exponential holding times: starting from state i time, T_i, until process leaves i has exponential distribution, rate denoted v_i.
- Sequence of states visited, $Y_0, Y_1, Y_2, \ldots$ is Markov chain – transition matrix has $\mathbf{P}_{ii} = 0$. Y sometimes called **skeleton**.
- **Communicating classes** defined for skeleton chain.
- Usually assume chain has 1 communicating class.
- Periodicity irrelevant because of continuity of exponential distribution.

Basic results and examples

- Instantaneous transition rates from i to j:

$$q_{ij} = v_i \mathbf{P}_{ij}$$

- Kolmogorov backward equations:

$$\mathbf{P}'_{ij}(t) = \sum_{k \neq i} q_{ik} \mathbf{P}_{kj}(t) - v_i \mathbf{P}_{ij}(t)$$

- Kolmogorov forward equations:

$$\mathbf{P}'_{ij}(t) = \sum_{k \neq j} q_{kj} \mathbf{P}_{ik}(t) - v_j \mathbf{P}_{ij}(t)$$

- For strongly recurrent chains with a single communicating class:

$$\mathbf{P}_{ij}(t) \to \pi_j$$

- Stationary initial probabilities π_i satisfy:

$$v_j \pi_j = \sum_{k \neq j} q_{kj} \pi_k$$

More basic results

- Transition probabilities given by

$$\mathbf{P}(t) = e^{\mathbf{R}t}$$

where $\mathbf{R}$ has entries

$$\mathbf{R}_{ij} = \begin{cases} q_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

- Process is a **Birth and Death** process if

$$\mathbf{P}_{ij} = 0 \text{ if } |i - j| > 1$$

- In this case we write λ_i for the instantaneous "birth" rate:

$$P(Y(t+h) = i+1 | Y_t = i) = \lambda_i h + o(h)$$

and μ_i for the instantaneous "death" rate:

$$P(Y(t+h) = i-1 | Y_t = i) = \mu_i h + o(h)$$

More basic results

- We have

$$q_{ij} = \begin{cases} 0 & |i-j| > 1 \\ \lambda_i & j = i+1 \\ \mu_i & j = i-1 \end{cases}$$

- If all $\mu_i = 0$ then process is a **pure birth** process.
- If all $\lambda_i = 0$ a **pure death** process.
- Birth and Death process have stationary distribution

$$\pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}\right)}$$

- Necessary condition for existence of π is

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} < \infty$$

Chapman-Kolmogorov Equations

- If X a Markov Chain with stationary transitions then

$$
\begin{aligned}
P(X(t+s) = k|X(0) = i)& \\
&= \sum_j P(X(t+s) = k, X(t) = j|X(0) = i) \\
&= \sum_j P(X(t+s) = k|X(t) = j, X(0) = i) \\
&\qquad \times P(X(t) = j|X(0) = i) \\
&= \sum_j P(X(t+s) = k|X(t) = j)P(X(t) = j|X(0) = i) \\
&= \sum_j P(X(s) = k|X(0) = j)P(X(t) = j|X(0) = i)
\end{aligned}
$$

- This shows the Chapman-Kolmogorov equations:

$$\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s) = \mathbf{P}(s)\mathbf{P}(t).$$

Extending the Markov Property

- Now consider the chain starting from i and let T_i be the first t for which $X(t) \neq i$.
- Then T_i is a stopping time.
- Technically: for each t:

$$\{T_i \le t\} \in \mathcal{H}_t$$

- Then

$$\begin{aligned} P(T_i > t+s | T_i > s, X(0) = i) &= P(T_i > t+s | X(u) = i; 0 \le u \le s) \\ &= P(T_i > t | X(0) = i) \end{aligned}$$

  by the Markov property.
- Note: we actually are asserting a generalization of the Markov property: If f is some function on the set of possible paths of X then

$$\begin{aligned} \mathrm{E}(f(X(s+\cdot)) | X(u) = x(u), 0 \le u \le s) \\ = \mathrm{E}\left[f(X(\cdot)) | X(0) = x(s)\right] \\ = \mathrm{E}^{x(s)}\left[f(X(\cdot))\right] \end{aligned}$$

Extending the Markov Property

- The formula requires some sophistication to appreciate.
- In it, f is a function which associates a sample path of X with a real number.
- For instance,

$$f(x(\cdot)) = \sup\{t : x(u) = i, 0 \le u \le t\}$$

  is such a functional.
- Jargon: **functional** is a function whose argument is itself a function and whose value is a scalar.
- FACT: Strong Markov Property – for a stopping time T

$$\mathrm{E}\left[f\left\{X(T+\cdot)\right\}|\mathcal{F}_T\right] = \mathrm{E}^{X(T)}\left[f\left\{X(\cdot)\right\}\right]$$

  with suitable fix on event $T = \infty$.
- Conclusion: given $X(0) = i$, T_i has memoryless property so T_i has an exponential distribution.
- Let v_i be the rate parameter.

Embedded Chain: Skeleton

- Let $T_1 < T_2 < \cdots$ be the stopping times at which transitions occur.
- Then $X_n = X(T_n)$.
- Sequence X_n is a Markov chain by the strong Markov property.
- That $\mathbf{P}_{ii} = 0$ reflects fact that $P(X(T_{n+1}) = X(T_n)) = 0$ by design.
- As before we say $i \rightsquigarrow j$ if $\mathbf{P}_{ij}(t) > 0$ for some t.
- It is fairly clear that $i \rightsquigarrow j$ for the $X(t)$ if and only if $i \rightsquigarrow j$ for the embedded chain X_n.
- We say $i \leftrightarrow j$ if $i \rightsquigarrow j$ and $j \rightsquigarrow i$.

Instantaneous Transition Rates

- Now consider
$$P(X(t+h)=j|X(t)=i,\mathcal{H}_t)$$
- Suppose the chain has made n transitions so far so that $T_n < t < T_{n+1}$.
- Then the event $X(t+h)=j$ is, except for possibilities of probability $o(h)$ the event that
$$t < T_{n+1} \le t+h \text{ and } X_{n+1}=j$$
- The probability of this is
$$(v_i h + o(h))\mathbf{P}_{ij} = v_i \mathbf{P}_{ij} h + o(h)$$

Kolmogorov's Equations

- The Chapman-Kolmogorov equations are

$$\mathbf{P}(t+h) = \mathbf{P}(t)\mathbf{P}(h)$$

- Subtract $\mathbf{P}(t)$ from both sides, divide by h and let $h \to 0$.
- Remember that $\mathbf{P}(0)$ is the identity.
- We find

$$\frac{\mathbf{P}(t+h) - \mathbf{P}(t)}{h} = \frac{\mathbf{P}(t)(\mathbf{P}(h) - \mathbf{P}(0))}{h}$$

  which gives

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{P}'(0)$$

- The Chapman-Kolmogorov equations can also be written

$$\mathbf{P}(t+h) = \mathbf{P}(h)\mathbf{P}(t)$$

- Now subtracting $\mathbf{P}(t)$ from both sides, dividing by h and letting $h \to 0$ gives

$$\mathbf{P}'(t) = \mathbf{P}'(0)\mathbf{P}(t)$$

Instantaneous Transition Rates

- Look at these equations in component form:

$$\mathbf{P}'(t) = \mathbf{P}'(0)\mathbf{P}(t)$$

becomes

$$\mathbf{P}'_{ij}(t) = \sum_k \mathbf{P}'_{ik}(0)\mathbf{P}_{kj}(t)$$

- For $i \neq k$ our calculations of instantaneous transition rates gives

$$\mathbf{P}'_{ik}(0) = v_i \mathbf{P}_{ik}$$

- For $i = k$ we have

$$P(X(h) = i | X(0) = i) = e^{-v_i h} + o(h)$$

($X(h) = i$ either means $T_i > h$ which has probability $e^{-v_i h}$ or there have been two or more transitions in $[0, h]$, a possibility of probability $o(h)$.)

- Thus

$$\mathbf{P}'_{ii}(0) = -v_i$$

Backward Equations

- Let $\mathbf{R}$ be the matrix with entries

$$\mathbf{R}_{ij} = \begin{cases} q_{ij} \equiv v_i \mathbf{P}_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

  $\mathbf{R}$ is the **infinitesimal generator** of the chain.

- Thus

$$\mathbf{P}'(t) = \mathbf{P}'(0)\mathbf{P}(t)$$

  becomes

$$\begin{aligned} \mathbf{P}'_{ij}(t) &= \sum_k \mathbf{R}_{ik} \mathbf{P}_{kj}(t) \\ &= \sum_{k \neq i} q_{ik} \mathbf{P}_{kj}(t) - v_i \mathbf{P}_{ij}(t) \end{aligned}$$

- Called **Kolmogorov's backward equations**.

Forward equations

- On the other hand

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{P}'(0)$$

becomes

$$\begin{aligned}\mathbf{P}'_{ij}(t) &= \sum_k \mathbf{P}_{ik}(t)\mathbf{R}_{kj} \\ &= \sum_{k \neq j} q_{kj}\mathbf{P}_{ik}(t) - v_j \mathbf{P}_{ij}(t)\end{aligned}$$

- These are **Kolmogorov's forward equations**.
- Remark: When the state space is infinite the forward equations may not be justified.
- In deriving them we interchanged a limit with an infinite sum; the interchange is always justified for the backward equations but not for forward.

Example

- Example: $S = \{0, 1\}$.
- Then

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

  and the chain is otherwise specified by v_0 and v_1.
- The matrix $\mathbf{R}$ is

$$\mathbf{R} = \begin{bmatrix} -v_0 & v_0 \\ v_1 & -v_1 \end{bmatrix}$$

Example continued

The backward equations become

$$\begin{aligned}
\mathbf{P}_{00}'(t) &= v_0\mathbf{P}_{10}(t) - v_0\mathbf{P}_{00}(t)\\
\mathbf{P}_{01}'(t) &= v_0\mathbf{P}_{11}(t) - v_0\mathbf{P}_{01}(t)\\
\mathbf{P}_{10}'(t) &= v_1\mathbf{P}_{00}(t) - v_1\mathbf{P}_{10}(t)\\
\mathbf{P}_{11}'(t) &= v_1\mathbf{P}_{01}(t) - v_1\mathbf{P}_{11}(t)
\end{aligned}$$

while the forward equations are

$$\begin{aligned}
\mathbf{P}_{00}'(t) &= v_1\mathbf{P}_{01}(t) - v_0\mathbf{P}_{00}(t)\\
\mathbf{P}_{01}'(t) &= v_0\mathbf{P}_{00}(t) - v_1\mathbf{P}_{01}(t)\\
\mathbf{P}_{10}'(t) &= v_1\mathbf{P}_{11}(t) - v_0\mathbf{P}_{10}(t)\\
\mathbf{P}_{11}'(t) &= v_0\mathbf{P}_{10}(t) - v_1\mathbf{P}_{11}(t)
\end{aligned}$$

Example

- Add v_1 times first and v_0 times third backward equations to get

$$v_1 \mathbf{P}'_{00}(t) + v_0 \mathbf{P}'_{10}(t) = 0 \text{ so } v_1 \mathbf{P}_{00}(t) + v_0 \mathbf{P}_{10}(t) = c.$$

- Put $t = 0$ to get $c = v_1$.
- This gives

$$\mathbf{P}_{10}(t) = \frac{v_1}{v_0} \{1 - \mathbf{P}_{00}(t)\}$$

- Plug this back in to the first equation and get

$$\mathbf{P}'_{00}(t) = v_1 - (v_1 + v_0)\mathbf{P}_{00}(t)$$

- Multiply by $e^{(v_1+v_0)t}$ and get

$$\left\{ e^{(v_1+v_0)t} \mathbf{P}_{00}(t) \right\}' = v_1 e^{(v_1+v_0)t}$$

  which can be integrated to get

$$\mathbf{P}_{00}(t) = \frac{v_1}{v_0 + v_1} + \frac{v_0}{v_0 + v_1} e^{-(v_1+v_0)t}$$

Use of Linear Algebra

- Alternative calculation:

$$\mathbf{R} = \begin{bmatrix} -v_0 & v_0 \\ v_1 & -v_1 \end{bmatrix}$$

can be written as

$$\mathbf{M}\boldsymbol{\Lambda}\mathbf{M}^{-1}$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & v_0 \\ 1 & -v_1 \end{bmatrix} \quad \mathbf{M}^{-1} = \begin{bmatrix} \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \\ \frac{1}{v_0+v_1} & \frac{-1}{v_0+v_1} \end{bmatrix}$$

and

$$\boldsymbol{\Lambda} = \begin{bmatrix} 0 & 0 \\ 0 & -(v_0+v_1) \end{bmatrix}$$

Matrix Exponentials

- Then

$$e^{\mathbf{R}t} = \sum_0^\infty \mathbf{R}^n t^n/n! = \sum_0^\infty \left(\mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1}\right)^n \frac{t^n}{n!} = \mathbf{M}\left(\sum_0^\infty \mathbf{\Lambda}^n \frac{t^n}{n!}\right)\mathbf{M}^{-1}$$

- Now

$$\sum_0^\infty \mathbf{\Lambda}^n \frac{t^n}{n!} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(v_0+v_1)t} \end{bmatrix}$$

so we get

$$\begin{aligned}\mathbf{P}(t) = e^{\mathbf{R}t} &= \mathbf{M}\begin{bmatrix} 1 & 0 \\ 0 & e^{-(v_0+v_1)t} \end{bmatrix}\mathbf{M}^{-1} \\ &= \mathbf{P}^\infty - \frac{e^{-(v_0+v_1)t}}{v_0+v_1}\mathbf{R}\end{aligned}$$

where

$$\mathbf{P}^\infty = \begin{bmatrix} \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \\ \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \end{bmatrix}$$

Linear Algebra

Theorem

If $\mathbf{A}$ *is a* $n \times n$ *matrix then there are matrices* $\mathbf{D}$ *and* $\mathbf{N}$ *such that* $\mathbf{A} = \mathbf{D} + \mathbf{N}$ *and*

1. $\mathbf{D}$ *is diagonalizable:*
$$\mathbf{D} = \mathbf{M}\boldsymbol{\Lambda}\mathbf{M}^{-1}$$
for some invertible $\mathbf{M}$ *and diagonal* $\boldsymbol{\Lambda}$.
2. $\mathbf{N}$ *is* **nilpotent**: *there is* $r \le n$ *such that* $\mathbf{N}^r = \mathbf{0}$.
3. $\mathbf{N}$ *and* $\mathbf{D}$ *commute:* $\mathbf{N}\mathbf{D} = \mathbf{D}\mathbf{N}$.

In this case

$$\begin{aligned}\mathbf{A}^k &= (\mathbf{D} + \mathbf{N})^k \\ &= \sum_{j=0}^{k} \binom{k}{j} \mathbf{D}^{k-j}\mathbf{N}^j\end{aligned}$$

More Matrix Exponentials

Thus

$$
\begin{aligned}
e^{\mathbf{A}} = \sum_{k=0}^{\infty} \mathbf{A}^k/k! &= \sum_{k=0}^{\infty} (\mathbf{D}+\mathbf{N})^k/k! \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{1}{(k-j)!j!}\mathbf{D}^{k-j}\mathbf{N}^j \\
&= \sum_{j=0}^{r-1} \frac{\mathbf{N}^j}{j!} \sum_{k=j}^{\infty} \frac{\mathbf{D}^{k-j}}{(k-j)!} \\
&= \sum_{j=0}^{r-1} \frac{\mathbf{N}^j}{j!} \sum_{k=0}^{\infty} \frac{\mathbf{D}^{k}}{k!} \\
&= e^{\mathbf{D}} \sum_{j=0}^{r-1} \frac{\mathbf{N}^j}{j!}
\end{aligned}
$$

Stationary Initial Distributions

- Notice: rows of $\mathbf{P}^\infty$ are a stationary initial distribution. If rows are π then

$$\mathbf{P}^\infty = \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \pi \equiv \mathbf{1}\pi$$

so

$$\pi \mathbf{P}^\infty = (\pi \mathbf{1})\pi = \pi$$

Moreover

$$\pi \mathbf{R} = \mathbf{0}$$

Fact: $\pi_0 = v_1/(v_0 + v_1)$ is long run fraction of time in state 0.
Fact:

$$\frac{1}{T}\int_0^T f(X(t))dt \to \sum_j \pi_j f(j)$$

Ergodic Theorem in continuous time.

Potential Pathologies

- Suppose that for each k you have a sequence

$$T_{k,1}, T_{k,2}, \cdots$$

such that all T_{ij} are independent exponential random variables and T_{ij} has rate parameter λ_j. We can use these times to make a Markov chain with state space $S = \{1, 2, \ldots\}$:
Start the chain in state 1. At time $T_{1,1}$ move to 2, $T_{1,2}$ time units later move to 3, etc. Chain progresses through states in order 1,2,....
We have

$$v_i = \lambda_i$$

and

$$\mathbf{P}_{ij} = \begin{cases} 0 & j \neq i+1 \\ 1 & j = i+1 \end{cases}$$

Does this define a process?

Pathologies Continued

- Depends on $\sum \lambda_i^{-1}$.
- Case 1: if $\sum \lambda_i^{-1} = \infty$ then

$$P(\sum_1^\infty T_{1,j} = \infty) = 1$$

(converse to Borel Cantelli) and our construction defines a process $X(t)$ for all t.

- Case 2: if $\sum \lambda_j^{-1} < \infty$ then for each k

$$P(\sum_{j=1}^\infty T_{kj} < \infty) = 1$$

In this case put $T_k = \sum_{j=1}^\infty T_{kj}$.

- Our definition above defines a process $X(t)$ for $0 \le t < T_1$.

Explosive paths

- We put $X(T_1) = 1$ and then begin the process over with the set of holding times $T_{2,j}$.
- This defines X for $T_1 \le t < T_1 + T_2$.
- Again we put $X(T_2) = 1$ and continue the process.
- Result: X is a Markov Chain with specified transition rates.

Re-entry from Infinity

- Problem: what if we put $X(T_1) = 2$ and continued?
- What if we used probability vector $\alpha_1, \alpha_2, \ldots$ to pick a value for $X(T_1)$ and continued?
- All yield Markov Processes with the same infinitesimal generator $\mathbf{R}$.
- Point of all this: gives example of non-unique solution of differential equations!

Birth and Death Processes

- Consider a population of $X(t) = i$ individuals.
- Suppose in next time interval $(t, t+h)$ probability of population increase of 1 (called a birth) is $\lambda_i h + o(h)$ and probability of decrease of 1 (death) is $\mu_i h + o(h)$.
- Jargon: X is a birth and death process.
- Special cases:
  - All $\mu_i = 0$; called a **pure birth** process.
  - All $\lambda_i = 0$ (0 is absorbing): **pure death** process.
  - $\lambda_n = n\lambda$ and $\mu_n = n\mu$ is a **linear** birth and death process.
  - $\lambda_n \equiv 1$, $\mu_n \equiv 0$: Poisson Process.
  - $\lambda_n = n\lambda + \theta$ and $\mu_n = n\mu$ is a **linear** birth and death process with immigration.

Applications

- Cable strength: Cable consists of n fibres.
- $X(t)$ is number which have *not* failed up to time t.
- Pure death process: μ_i will be large for small i, small for large i.
- Chain reactions. $X(t)$ is number of free neutrons in lump of uranium.
- Births produced as sum of: spontaneous fission rate (problem — I think each fission produces 2 neutrons) plus rate of collision of neutron with nuclei.
- Ignore: neutrons leaving sample and decay of free neutrons.
- Get $\lambda_n = n\lambda + \theta$
- At least in early stages where decay has removed a negligible fraction of atoms.

Stationary initial distributions

- As in discrete time an initial distribution is probability vector π with

$$P(X(0) = i) = \pi_i$$

- An initial distribution π is **stationary** if

$$\pi = \pi \mathbf{P}(t)$$

or

$$P(X(t) = i) = \pi_i$$

for all $t \geq 0$.

- If so take derivative wrt t to get

$$0 = \pi \mathbf{P}'(t)$$

or

$$\pi \mathbf{R} = 0$$

Stationary Initial Distributions

- Conversely: if
$$\pi\mathbf{R} = 0$$
then
$$\begin{aligned}\pi\mathbf{P}(t) &= \pi e^{\mathbf{R}t}\\ &= \pi\left(\mathbf{I} + \mathbf{R}t + \mathbf{R}^2t^2/2 + \cdots\right)\\ &= \pi\end{aligned}$$
So a probability vector π such that
$$\pi\mathbf{R} = 0$$
is a stationary initial distribution.
- NOTE: π is a left eigenvector of $\mathbf{P}(t)$.
- Perron-Frobenius theorem asserts that 1 is the largest (in modulus) eigenvalue of $\mathbf{P}(t)$, that this eigenvalue has multiplicity 1, that the corresponding eigenvector has all positive entries.
- So: can prove every row of $\mathbf{P}(t)$ converges to π.

Conditions for stationary initial distribution

1. $v_n = \lambda_n + \mu_n$.
2. $P_{n,n+1} = \lambda_n / v_n = 1 - P_{n,n-1}$.
3. From $\pi \mathbf{R} = 0$:
$$v_n \pi_n = \lambda_{n-1}\pi_{n-1} + \mu_{n+1}\pi_{n+1}$$
4. Start at $n = 0$:
$$\lambda_0 \pi_0 = \mu_1 \pi_1$$
so $\pi_1 = (\lambda_0/\mu_1)\pi_0$.
5. Now look at $n = 1$.
$$(\lambda_1 + \mu_1)\pi_1 = \lambda_0\pi_0 + \mu_2\pi_2$$
6. Solve for π_2 to get
$$\pi_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\pi_0$$
And so on.
7. Then use $\sum \pi_n = 1$.

Stationary initial distribution of skeleton

- Relation of π to stationary initial distribution of skeleton chain.
- Let α be stationary initial dist of skeleton.
- Heuristic: fraction of time in state j proportional to fraction of skeleton visits to state j times average time spent in state j:

$$\pi_j \propto \alpha_j \times (1/v_j)$$