Diffusions

Richard Lockhart

Simon Fraser University

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Purposes of Today's Lecture

- Describe diffusion modelling assumptions.
- Derive Kolmogorov's backwards equation.
- Deduce partial differential equation.
- Discuss special case of heat equation.

Diffusions

- Next natural generalization: continuous time, continuous state space Markov Chains: $S = \mathbb{R}$.
- Study processes $\{X(t); t \geq 0\}$ where each $X(t)$ is real valued and $t \mapsto X(t)$ is continuous.
- Idea for diffusions: model short time behaviour: given $\mathcal{H}_t$

$$X(t+h) = X(t) + \mu(X(t))h + \sigma(X(t))\sqrt{h}\epsilon + o(h)$$

  where ϵ is a standard normal variable and $\mu(\cdot)$ and $\sigma(\cdot)$ are model specified functions.
- Meaning: the equation is an assertion about the approximate distribution of $X(t+h) - X(t)$.
- Chapman-Kolmogorov. Let $f(t,x,y)$ be the density of $X(t)$ at y given $X(0) = x$. Then

$$f(t+s,x,y) = \int_{-\infty}^{\infty} f(s,x,z)f(t,z,y)dz$$

Kolmogorov's Backward Equations

- Informal presentation.
- Let h be small. Assume that

$$f(h,x,y) \approx \frac{\exp\{-(y-x-\mu(x)h)^2/(2\sigma^2(x)h)\}}{\sqrt{2\pi h}\sigma(x)}$$

- That is, we assume that $X(h)$ is, given $X(0)$, approximately normally distributed with mean $X(0)+\mu(X(0))h$ and variance $h\sigma^2(X(0))$.
- Then

$$f(t+h,x,y) = \int f(h,x,z)f(t,z,y)dz\,.$$

Backward Equations Continued

- Make the change of variables

$$u = \frac{z - x - \mu(x)h}{\sigma(x)\sqrt{h}} \text{ and } du = \frac{dz}{\sigma(x)\sqrt{h}}$$

to find

$$f(t+h,x,y) = \int \frac{e^{-u^2/2}}{\sqrt{2\pi}} f(t, x+\mu(x)h+\sigma(x)u\sqrt{h}, y)du$$

- Suppose f is twice differentiable wrt x: Taylor expansion:

$$\begin{aligned} f(t,&x+\mu(x)h+\sigma(x)u\sqrt{h},y) \\ &= f(t,x,y) + \frac{\partial f}{\partial x}(\mu(x)h+\sigma(x)u\sqrt{h}) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu(x)h+\sigma(x)u\sqrt{h})^2 + o(h) \end{aligned}$$

Partial derivatives evaluated at (t, x, y).

Backward Equations Continued

- Put these into Chapman Kolmogorov and use

$$\int u \frac{e^{-u^2/2}}{\sqrt{2\pi}} du = 0$$

to get

$$f(t+h,x,y) = f(t,x,y) + \mu(x)h\frac{\partial f}{\partial x} + \frac{\sigma^2(x)}{2}h\frac{\partial^2 f}{\partial x^2} + o(h).$$

- Move $f(t,x,y)$ to other side, divide by h and take limit:

$$\frac{\partial f}{\partial t} = \mu(x)\frac{\partial f}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2 f}{\partial x^2}.$$

- This differential equation is called a diffusion equation.
- Special case: $\mu \equiv 0$ and $\sigma \equiv 1$ gives the heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$$

Solving the heat equation

- Consider simpler case: Compute

$$H(t,x) \equiv \mathrm{E}^x\left[\phi(X_t)\right]$$

  for some nice function ϕ.
- Multiply Chapman-Kolmogorov equations by $\phi(y)$ and integrate dy.
- Discover H also solves heat equation:

$$\frac{\partial H}{\partial t} = \frac{1}{2}\frac{\partial^2 H}{\partial x^2}$$

  with initial condition:

$$H(0,x) = \phi(x)$$

- Solved by taking Fourier transforms. Get:

$$H(t,x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/(2t)} \phi(y) dy$$

- Interpretation: X_t is $N(x,t)$.
- Another interpretation: $f(t,x,y)$ is normal density centered at X with standard deviation $\sqrt{t}$.