

The Lions Gate Bridge

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Purposes of Today's Lecture

- Illustrate modelling process.
- Show some probabilistic notation.
- Show potential use of inequalities.
- Introduce independence, overdispersion, Markov's inequality.



Traffic Loading on Lions Gate Bridge

- Idea: want to know how strong bridge needs to be.
- Compute: load x such that
Expected time to first exceedance of load x is 100 years.
- Method uses:
 - 1 modelling assumptions.
 - 2 conservative modelling; to replace random variable of interest with stochastically larger quantity.
 - 3 moment generating functions; Markov's inequality — compute upper bound on x .



The Lions Gate Bridge



Facts about the Lions Gate Bridge

- Built 1936-38 for \$6M.
- 3 spans: 614, 1550, 614 feet long.
- Originally 2 lanes now 3.
- Originally toll bridge built by developers.
- See

<http://www.b-t.com/menu/project/rehabilitationandseismicretrofit/Page>
at Buckland and Taylor web site for engineering info.



The Lions Gate Bridge



Process of Interest

- Think of bridge as rectangle.
- Co-ordinates: x runs from 0 to length L_B of bridge, y runs from 0 to width W_B of bridge.
- Define:

$$Z(x, y, t) = \text{load on bridge at } (x, y) \text{ at time } t$$

- General quantity of interest at time t : total load or other force on segment of bridge:

$$\int_{xy} Z(x, y, t) w(x, y) dx dy$$

- Example w : load in strip across bridge between x_1 and x_2 feet out from south side on central span

$$W(t, x_1, x_2) = \int_{x_1}^{x_2} \int_0^{W_B} Z(x, y, t) dy dx$$



Quantity of concern to engineers

- Quantity of concern to engineers:

$$M_T(L) \equiv \max_{t \in [0, T]} \max_{0 \leq x_1 \leq L_B - L} W(t, x_1, x_1 + L)$$

- First modelling assumption. Years $1, \dots, T$ are iid.
- So:

$$P(M_T(L) \leq y) = P(M_1(L) \leq y)^T$$

- So: years to first exceedance of level y has geometric distribution with probability of success

$$P(M_1(L) > y)$$

- Find y so this last is $1/100$; expected value of geometric is 100.
- Call this the 100 year return time load.



Next modelling consideration.

- Two kinds of loads: static and dynamic.
- Consider only static not dynamic loading.
- Observation: static loading much higher when traffic stopped than not.
- So: define N to be number of traffic stoppages in year.
- Let $M_{1,n}(L)$ be worst load over segment of length L during n th of N stoppages.
- Idea

$$P(M_1(L) > y) = P(\max_{1 \leq n \leq N} M_{1,n}(L) > y)$$

Treat $M_{1,n}(L)$ as iid *given* N .



Conditioning

- Next: Evaluate $P(M_1(L) > y)$ by conditioning.
- Shorten notation:

$$M = \max_{1 \leq n \leq N} X_n$$

where X_i iid, cdf F , survival ftn $S = 1 - F$.

$$\begin{aligned} P(M \leq y) &= E\{P(M \leq y|N)\} \\ &= E[\{1 - S(y)\}^N] \\ &= \phi[\log\{1 - S(y)\}] \end{aligned}$$

where

$$\phi(t) = E\left(e^{tN}\right)$$

is the moment generating function of N .



Solve for y

- Comment: ϕ is monotone increasing.
- So: if $S(y) \leq g(y)$ then

$$P(M > y) \leq 1 - \phi[\log\{1 - g(y)\}]$$

and solving

$$\phi[\log\{1 - g(y)\}] = 0.99$$

gives larger solution than

$$P(M \leq y) = 0.99$$

- Remaining steps:
 - 1 Model for N .
 - 2 Model / upper bound for S .



Modelling N :

- Simplest idea: Poisson process of accidents.
- So N has $\text{Poisson}(\lambda)$ dist for some λ .
- Then

$$\phi(t) = \sum e^{-\lambda} \frac{(\lambda e^t)^n}{n!}$$

which is

$$\phi(t) = \exp\{\lambda(e^t - 1)\}$$

- Criticisms: No allowance for variation in traffic densities, weather, etc from year to year.



More sophisticated assumption

- Potentially better assumption.
- N is overdispersed Poisson, say, Negative Binomial:

$$P(N = k) = \binom{r + k - 1}{k} p^r (1 - p)^k \quad k = 0, 1, \dots$$

- This makes

$$E(e^{tX}) = \frac{p^r}{\{1 - (1 - p)e^t\}^r}$$

- Idea: for Poisson $\sigma = \sqrt{\mu}$.
- For Negative Binomial $\mu = r(1 - p)/p$ and

$$\sigma = \sqrt{r(1 - p)/p^2} = \sqrt{1/p} \sqrt{\mu} > \sqrt{\mu}$$

- Idea: use of overdispersed variable makes for longer tails relative to mean.



Use of Upper Bounds

- Now we need to model / bound the survival function S .
- Stoppage lasts random time T .
- During that time traffic builds up behind stoppage; cars jam together.
- Worst section of length L found by sliding window along line of stopped cars to find maximum.
- Notional model (not the way we did it):
- Model vehicles arriving at end of queue.
- Might use Poisson Process.
- Each vehicle has random mass, length, distribution of load along length.
- Random gaps between vehicles.
- Just before traffic starts to move again: look for heaviest segment of length L in stoppage.



Problems

- 1 different kinds of stoppage: $\#$ lanes, direction of flow, location on bridge, cars trickle past?
 - 2 hard to deal with supremum over all segments of length L .
 - 3 specify joint law of mass, length, distribution of load along single vehicle.
- Digression to method we didn't use:
 - Model length of stoppage T with density g .
 - Model N , number of vehicles arriving at end of stoppage, given T as $\text{Poisson}(\lambda T)$.
 - Assume next vehicle arriving picked at random; joint density $h(w, l)$ of weight, length. W_i, L_i values for i th arrival.



Physical Length of stoppage

- Assume load distributed evenly along length of vehicle.
- Final length of line at end of stoppage is

$$L_T \equiv \sum_{i=1}^N L_i.$$

- Can compute mean, variance, generating function of L_T ?

$$\begin{aligned} E\{\exp(sL)\} &= E[E\{\exp(sL)|N\}] \\ &= E\left([E\{\exp(sL_i)\}]^N\right) \\ &= \phi_N[\log\{\phi_L(s)\}] \end{aligned}$$

Here each ϕ is a moment generating function.



Compound Poisson processes

- Method of analysis for a compound Poisson Process.
- Can use the mgf of L to compute distribution of L by inversion of Laplace transform.
- Problem: how to scan for maximum load?



Discretization

- Simplify problem: discretize and bound.
- If h is small and $X(s)$ some process then

$$\sup_{0 \leq t \leq T-\tau} \int_t^{t+\tau} X(s) ds$$

is close to

$$\max_k \int_{kh}^{kh+\tau} X(s) ds$$

- We took h to be 50 feet and considered $\tau = 50n$ feet.
- Switch from thinking about length of stoppage in time to length of stoppage in multiples of h .
- Let N_i be number of segments of length h building up on bridge during stoppage
- Let $X_j; j = 1, \dots, N_i$ be the loads on the consecutive segments.



Upper Bounds

- So: our interest is in

$$\max\{X_r + \cdots X_{r+n-1}; 1 \leq r \leq N_i + 1 - n\}$$

- Upper bound on survival function?
- Define

$$U_r = X_r + \cdots X_{r+n-1}$$

- Argue that

$$1 - P(\max_{1 \leq r \leq m} U_r > y) \leq 1 - \prod_{1 \leq r \leq m} \{1 - P(U_r > y)\}$$

for any m .

- Rationale: Values of U_r are *positive orthant dependent*.
- (Large values of one U suggest large values of adjacent U .)



Modelling Details

- So now:

$$P(\max\{U_r; 1 \leq r \leq N_i\} > y) \leq 1 - \phi_{N_i}[\log\{1 - S_U(y)\}]$$

- Model law of X_i by considering possible loading patterns by cars, trucks, buses.
- We took cars to be fixed length and weight.
- Same for buses.
- Trucks had fixed length, weight uniform on 12 to 40 tons.
- Computed moment generating function of an X :

$$\phi_X(t) = E(e^{tX})$$



Markov's inequality

- Final step. Need to compute S_U .
- Instead use Markov's inequality:

$$P(X \geq x) \leq \frac{E\{g(X)\}}{g(x)}$$

for any increasing positive g .

- Choose $g(\cdot) = \exp(h\cdot)$.
- So:

$$\begin{aligned} S_U(y) &\leq \frac{E(e^{hU})}{\exp(hy)} \\ &= \frac{\{\phi_X(h)\}^n}{\exp(hy)} \end{aligned}$$

where

$$\phi_X(h) = E\{\exp(hX)\}$$



Optimize Markov's inequality

- These can be assembled to give a bound depending on h .
- Then: minimize over $h > 0$ to find good bound.
- I wrote FORTRAN code to do this in 1975 for my summer job with Jim Zidek.
- He was consulting with Frank Navin for Peter Buckland.
- Zidek, James V., Navin, Francis P. D. and Lockhart, R. A. (1979). Statistics of extremes: An alternate method with application to bridge design codes. *Technometrics*, **21**, 185–191.



Summary of Today's Lecture

- Notice big picture of modelling process.
- Define variables, introduce notation.
- Formulate problem (here as one of bounding return period).
- Make assumptions about joint and conditional distributions.
- Use conditioning.
- Use independence, overdispersion, dependence notions.
- Use Markov's inequality or numerical inversion of Laplace transform.
- Notice wealth of detailed modelling assumptions

