Renewal Processes

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Purposes of Today's Lecture

- Define Renewal Processes.
- Define Regeneration Times.

Renewal Theory

- Basic idea: study processes where after random time everything starts over at the beginning.
- Example: M/G/1 queue starts over every time the queue empties.
- Begin with **renewal process**:
- Have counting process $N(t)$.
- Times between arrivals are $T_1, T_2, \cdots$
- Time of nth arrival is

$$S_n = \sum_{i=1}^{n} T_i$$

- If arrival times iid with distribution F call N a renewal process.
- Poisson process is example with F an exponential cdf.

The counting process

- Define $N(t)$ = number of renewals by time t.
- So $N(t) = k$ if and only if

$$S_k \le t < S_{k+1}$$

- So:

$$\begin{aligned} P(N(t) = k) &= P(S_k \le t < S_{k+1}) \\ &= P(S_k \le t) - P(S_k \le t \cap S_{k+1} \le t) \\ &= P(S_k \le t) - P(S_{k+1} \le t) \end{aligned}$$

- Jargon: cdf of sum of k iid T_i is called **convolution**.

Basic principles

- In the long run the process forgets its starting time.
- Long run renewal rate is $1/\mu$ where μ is the expected lifetime of one X.
- Instantaneous renewal rate is eventually $1/\mu$. (Not conditional!)

Mean Values

- Mean values: define $m(t) = \mathrm{E}(N(t))$.

$$\begin{aligned} m(t) &= \mathrm{E}(N(t)) \\ &= \sum_k kP(N(t) = k) \\ &= \sum_k P(N(t) \ge k) \\ &= \sum_k P(S_k \le t) \end{aligned}$$

- Fact: m is finite.

Proof

- Find c so that $p = P(T_1 \le c) < 1$.
- Success: $T_i \le c$.
- Failure: $T_i > c$.
- $B = \#$ Successes $\sim$ Binomial(n, p).
- If $n - B > t/c$ then $S_n > t$.
- So

$$\begin{aligned}
P(S_n \le t) &\le P(B \ge n - t/c) \\
&= P(e^{\lambda B} \ge e^{\lambda(n-t/c)}) \\
&\le \frac{E(e^{\lambda B})}{e^{\lambda(n-t/c)}} \\
&= e^{t/c}(pe^{\lambda} + 1 - p)^n e^{-\lambda n} \\
&= e^{t/c}\left\{p + (1-p)e^{-\lambda}\right\}^n
\end{aligned}$$

- This is summable.

Compute m

- In fact compute m by conditioning on T_1:

$$\mathrm{E}(N(t)) = \mathrm{E}\left[\mathrm{E}(N(t)|T_1)\right]$$

- If $x > t$ and we are given $T_1 = x$ then $N(t) = 0$.
- If $x \le t$ and we are given $T_1 = x$ then $N(t)$ has the same law as

$$1 + N(t-x) \text{ so for } x \le t \; \mathrm{E}[N(t)|T_1 = x] = 1 + m(t-x)$$

- This makes

$$\mathrm{E}(N(t)|T_1) = \{1 + m(t - T_1)\} 1(T_1 \le t)$$

- Take expected values: **Renewal equation**

$$m(t) = F(t) + \mathrm{E}[m(t - T_1)1(T_1 \le t)]$$

- If F has density f

$$m(t) = F(t) + \int_0^t m(t-x)f(x)dx.$$

Basic renewal limit theorems

- Let $\mu = \mathrm{E}(T_1)$.
- First as $t \to \infty$:

$$N(t)/t \to 1/\mu$$

- Second: the elementary renewal theorem:

$$m(t)/t \to 1/\mu$$

- Note: not as easy to prove as it looks.
- Example: if $f(x) = 1(0 < x < 1)$ then renewal equation says, for $0 < t < 1$:

$$m(t) = t + \int_0^t m(t-x)dx = t + \int_0^t m(x)dx$$

Elementary renewal theorem

- Differentiate:

$$m'(t) = 1 + m(t)$$

or

$$\log(1 + m(t)) = t + c$$

Put $t = 0$ to find $c = 0$ and

$$m(t) = e^t - 1 \qquad \text{for } 0 < t < 1$$

- Not linear!
- For $1 < t < 2$:

$$\begin{aligned} m(t) &= 1 + \int_0^1 m(t-x)dx \\ &= 1 + \int_{t-1}^t m(u)du \end{aligned}$$

- Differentiate and solve to get

$$m(t) = e^{t-1}(1-t) + e^t - 1.$$

Regeneration

- Now consider a stochastic process with the property:
- There is a random time T such that:

$$P(T < \infty) = 1$$

and such that at time T the process starts over: the conditional distribution of the future given T and everything happening up to time T is the unconditional distribution of the process started at time 0.

- Called a **regeneration** (or **renewal**) time.
- Gives rise to sequence of times $T_1, T_2, \cdots$ which are iid.
- Let $N(t)$ denote number of renewals by time t.

Use of renewal theorems with regeneration times

- Associate to each cycle some random variable R_k, iid.
- Typically same function applied to path of process over one cycle.)
- Define

$$R(t) = \sum_{i=1}^{N(t)} R_i$$

- Basic facts:

$$\frac{R(t)}{t} = \frac{R(t)}{N(t)} \frac{N(t)}{t} \to \frac{\mathrm{E}(R_1)}{\mu}$$

and

$$\frac{\mathrm{E}[R(t)]}{t} = \frac{\mathrm{E}[R(t)]}{m(t)} \frac{m(t)}{t} \to \frac{\mathrm{E}(R_1)}{\mu}$$

Processes with regeneration times

1. Recurrent Markov chains
2. M/G/1 queue with input rate less than output rate.
3. G/M/1 queue with input rate less than output rate.

- Look at # 3: In each cycle think of B_i as busy time and I_i as idle time.
- Total length of cycle is $B_i + I_i$.
- Let $R(t)$ be amount of idle time up to time t.
- Get:

$$\frac{R(t)}{t} \to \frac{\mathrm{E}(I_1)}{\mathrm{E}(I_1 + B_1)}$$

and

$$\frac{\mathrm{E}[R(t)]}{t} \to \frac{\mathrm{E}(I_1)}{\mathrm{E}(I_1 + B_1)}$$

Can we compute the pieces?

- Number served from start of busy period to start of next busy period is N.
- $T_1, T_2, \cdots$ interarrival times for input.
- Total length of cycle is
$$\sum_{i=1}^{N} T_i$$
- Fact: N is a stopping time ($\{N = n\}$ is independent of $T_{n+1}, \cdots$.
- Wald's identity (added to homework):
$$\mathrm{E}\left[\sum_{i=1}^{N} T_i\right] = \mathrm{E}[N]\,\mathrm{E}[T_1]$$
- Note $\mathrm{E}[T_1] = \int t dG(t) \equiv 1/\lambda$.

Expected waiting time

- Compute $\mathrm{E}[N]$?
- N is number of transitions of Markov chain between visits to state 0.
- So $\pi_0 = 1/\mathrm{E}[N]$.
- That is
$$\mathrm{E}[N] = 1/(1-\beta)$$
- So expected cycle length is
$$\frac{1}{\lambda(1-\beta)}$$

Fraction of time in state k

- Ross presents following argument.
- Let P_k denote fraction of time system has k people in line.
- In steady state: transition rate from k to $k+1$ must balance reverse transition rate.
- Downward rate is $P_{k+1}\mu$. (Proportion of time in state $k+1$ times service rate.)
- Upward rate is average arrival rate times proportion of arrivals finding k in system or

$$\pi_k \lambda$$

- Get, for $k \geq 0$

$$P_{k+1}\mu = \pi_k \lambda$$

Or

$$P_{k+1} = \frac{\lambda}{\mu}(1-\beta)\beta^k$$

Since $\sum_0^\infty P_k = 1$ can solve for

$$P_0 = 1 - \lambda/\mu.$$

Summary of Conclusions

- Long run fraction of time system has k people in line or service is

$$P_k = \left(1 - \frac{\lambda}{\mu}\right)(1-\beta)\beta^{k-1} \text{ for } k \ge 1.$$

- Long fraction of time system is idle is

$$P_0 = \left(1 - \frac{\lambda}{\mu}\right) = \frac{\mathrm{E}(I_1)}{\mathrm{E}(I_1 + B_1)}.$$

- Expected cycle length is

$$\mathrm{E}(I_i + B_i) = \frac{1}{\lambda(1-\beta)} = \frac{\mu_G}{1-\beta}.$$

- So expected length of an idle period is

$$\mathrm{E}(I_i) = P_0 \mathrm{E}(I_i + B_i) = (1 - \lambda/\mu)\frac{1}{\lambda(1-\beta)} = \frac{1/\lambda - 1/\mu}{1-\beta}$$

- Expected length of a busy period is

$$\mathrm{E}(B_i) = \frac{1}{\lambda(1-\beta)} - \frac{1/\lambda - 1/\mu}{1-\beta} = \frac{1}{\mu(1-\beta)}.$$