Stochastic Differential Equations

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Purposes of Today's Lecture

- Motivate Stochastic Differential Equations.
- Describe Ito and Stratonovich integrals.

Stochastic Differential Equations

- Return to definition of diffusion: given $\mathcal{H}_t$

$$X(t+h) = X(t) + \mu(X(t))h + \sigma(X(t))\sqrt{h}\epsilon + o(h)$$

  where $\epsilon \sim N(0,1)$; $\mu(\cdot)$ and $\sigma(\cdot)$ are model specified functions.

- Use Brownian motion to give ϵ:

$$\begin{aligned} X(t+h) = X(t) &+ \mu(X(t))h \\ &+ \sigma(X(t))\left\{B(t+h) - B(t)\right\} + o(h) \end{aligned}$$

- Usually written in differential form $h = dt$:

$$dX_t = \mu(X(t))dt + \sigma(X(t))dB_t$$

- Interpretation is integral:

$$X_t = X_0 + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dB_s$$

- Meaning?

Riemann-Stieltjes Integrals

- F monotone increasing (right continuous, say) on $[0,t]$; f continuous on $[0,t]$:

$$\int_0^t f(s)F(ds)$$

is defined as follows.

- Fix N. Let $t_k = tk/N$ for $k = 0, 1, \ldots, N$.
- Put

$$\overline{I}_N = \sum_{k=1}^{n} \max\{f(u) : t_{k-1} \le u \le t_k\}\{F(t_k) - F(t_{k-1})\}$$

and

$$\underline{I}_N = \sum_{k=1}^{n} \min\{f(u) : t_{k-1} \le u \le t_k\}\{F(t_k) - F(t_{k-1})\}$$

Riemann-Stieltjes integrals continued

- Then

$$\int_0^t f(s)F(ds) = \lim_N \overline{I}_N = \lim_N \underline{I}_N$$

- If F is absolutely continuous then

$$\int_0^t f(s)F(ds) = \int_0^t f(s)F'(s)ds$$

which is an ordinary Riemann integral.

The integration problem

- Idea extends to F of "bounded variation" – difference of two monotone increasing G.
- Back to SDE: What is

$$\int_0^t \sigma(X(s))dB_s?$$

- Problem: bounded variation means

$$\sup_N \sum_k |F(t_k) - F(t_{k-1})| < \infty$$

- But

$$\lim_{N\to\infty} \sum_k |B(t_k) - B(t_{k-1})| = \infty$$

  because these are sums of N iid terms with mean proportional to $1/\sqrt{N}$.
- Consider example to see details of problem.

Stochastic Integrals

- What is

$$\int_0^t B_s dB_s?$$

NOT

$$B_t^2/2$$

- Two discrete approximations in spirit of Riemann Stieltjes:

$$I_{1,N} \equiv \sum_{k=0}^{N-1} B(t_k)\{B(t_{k+1}) - B(t_k)\}$$

and

$$I_{2,N} \equiv \sum_{k=0}^{N-1} B(t_{k+1})\{B(t_{k+1}) - B(t_k)\}$$

Stochastic Integrals Continued

- First has mean 0.
- Notice
$$I_{2,N} - I_{1,N} = \sum_{k=0}^{N-1} \{B(t_{k+1}) - B(t_k)\}^2$$
- If we multiply the kth term by N we get a χ_1^2 random variable so the difference is an average of N independent χ_1^2s.

Unbounded variation bounded quadratic variation

- So

$$I_{2,N} - I_{1,N} \to t$$

On the other hand

$$I_{2,N} + I_{1,N} = B_t^2 - B_0^2 = B_t^2$$

- Thus

$$I_{1,n} \to (B_t^2 - t)/2 \text{ and } I_{2,n} \to (B_t^2 + t)/2$$

- Use centered value of B in definition to make $B_t^2/2$ appear.
- The Ito integral

$$\int_0^t B_s dB_s = (B_t^2 - t)/2$$

is a match for our modelling tactic above.

- Centred version is Stratonovich integral.

Questions of interest

- Existence of solutions of SDEs?
- Calculus of stochastic integrals.

Example 1: Geometric Brownian Motion

- Consider $\mu(x) = \alpha x$ and $\sigma(x) = \beta x$ for $x > 0$, $\beta > 0$.
- Idea is change in X_t has mean and standard deviation proportional to X_t.
- So both constant in percentage terms.
- Solution of

$$dX_t = \alpha X_t dt + \beta X_t dB_t$$

is Geometric Brownian Motion:

$$X_t = X_0 \exp\left\{(\alpha - \beta^2/2)t + \beta B_t\right\}.$$

Example 2: Fisher Wright model

- Wright Fisher or Fisher Wright model of mutation.
- Idea is population of N individuals of genetic type A or a.
- Total number of genes is $2N$.
- Random pairing to form next generation: number of A genes has Binomial$(2N, p)$ distribution where p is fraction of current generation which is type A.
- BUT probability individual gene mutates A to a is α.
- AND probability individual gene mutates a to A is α'.
- Get discrete time Markov Chain: X_n is number of A in generation n; only change is to p in Binomial law. Given $X_n = i$

$$p = \frac{i}{2N}(1-\alpha) + \frac{2N-i}{2N}\alpha'.$$

Fisher-Wright in Continuous Time

- Mutation rates are $\alpha = \delta/(2N)$ and $\alpha' = \delta'/(2N)$.
- Let $N \to \infty$.
- $X_n(t)$ is proportion of population of type A at time $t = n/(2N)$ and X_t is the limit.
- Get
$$dX_t = \left\{-\delta X_t + \delta'(1 - X_t)\right\} dt + \sqrt{X_t(1 - X_t)} dB_t$$
- Fact: solution exists with $0 < X_t < 1$ for all t
- Fact: solution is Markov process with stationary initial distribution.

Example 3: Ornstein Uhlenbeck

- Ornstein Uhlenbeck model: velocity of Brownian particle
- Model velocity of particle (not position as in Brownian motion).
- Introduce friction proportional to velocity:

$$dV_t = -\alpha V_t dt + \sigma dB_t.$$

- Solution is

$$V_t = e^{-\alpha t}\left\{V_0 + \sigma \int_0^t e^{\alpha s} dB_s\right\}.$$

- This is a Gaussian process (joint distributions are normal).
- Its integral gives position.
- The process has a stationary initial distribution.

Stationary Initial Distributions

- If a stationary initial density π exists then

$$\pi(y) = \lim_{t\to\infty} f(t,x,y)$$

- In this case we may expect

$$\lim_{t\to\infty} \frac{\partial}{\partial t} f(t,x,y) = 0.$$

- The Kolmogorov **forward** equation involves this partial derivative.
- Back to Chapman Kolmogorov in general form.
- For Markov process which might not have stationary transitions: $f(s,t,x,y)$ is conditional density (at y) of $X(t)$ given $X(s) = x$.
- Then for $s < u < t$

$$f(s,t,x,y) = \int f(s,u,x,z) f(u,t,z,y) dz$$

Kolmogorov Forward Equation

- Replace s by 0, t by $t+h$ and u by t.
- Use fact $f(t,t+h,z,y)$ is approximately normal density with mean $\mu(z)h$ and variance $\sigma^2(z)h$:

$$\begin{aligned} f(t,t+h,z,y) &\approx \frac{1}{\sigma(z)\sqrt{2\pi h}} \exp\left(-\frac{(y-z-\mu(z)h)^2}{2\sigma^2(z)h}\right) \\ &= \frac{1}{\sigma(z)\sqrt{2\pi h}} \exp\left(-\frac{(z-y+\mu(z)h)^2}{2\sigma^2(z)h}\right) \end{aligned}$$

- Change variables to

$$u = \frac{z-y+\mu(y)h}{\sigma(y)\sqrt{h}}$$

- Notice $\mu(y), \sigma(y)$, not $\mu(z), \sigma(z)$.

Kolmogorov Forward Equation Continued

- After substitution expand

$$f(s,t,x,z)f(t,t+h,z,y)$$

  in powers of $\sqrt{h}$.
- Lengthy algebra ensues, smoke clears (with aid, for me, of Maple) to give

$$\frac{\partial}{\partial t}f(t,x,y) = -\frac{\partial}{\partial y}\mu(y)f(t,x,y) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\sigma^2(y)f(t,x,y)$$

- So stationary density π satisfies

$$\frac{\partial}{\partial y}\mu(y)\pi(y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\sigma^2(y)\pi(y)$$

Solving for the stationary distribution — Fisher Wright

- Equation becomes

$$\left(\delta'(1-y)-\delta y)\pi(y)\right)' = \frac{1}{2}\left(y(1-y)\pi(y)\right)''$$

- So for some constant c

$$(\delta'(1-y)-\delta y)\pi(y) = \frac{1}{2}\left(y(1-y)\pi(y)\right)' + c$$

- Now argue that at $y=0,1$ left hand side should vanish; not trivial.
- Or just try to find a solution with $\pi(0)=\pi(1)=0$ so $c=0$.
- This simplifies to

$$(\delta'(1-y)-\delta y)\pi(y) - (1-2y)\pi(y)/2 = y(1-y)\pi'(y)/2.$$

Applied to Fisher Wright

- Divide by $\pi(y)$ to get

$$\frac{(2\delta' - 1)(1-y) - (2\delta - 1)y}{y(1-y)} = \frac{\pi'(y)}{\pi(y)}.$$

- Integrate to get

$$\log \pi(y) = (2\delta' - 1)\ln(y) + (2\delta - 1)\ln(1-y) + c$$

or

$$\pi(y) = Cy^{2\delta' - 1}(1-y)^{2\delta - 1}$$

- This is a Beta$(2\delta', 2\delta)$ density giving C in terms of Gamma functions.

Ito Calculus

- For ordinary calculus: if $x(t)$ is a smooth (differentiable) function of t and $f(x,t)$ is continuously differentiable in both arguments then

$$\begin{aligned} df(x(t),t) =& f_x(x(t),t)dx(t) + f_t(x(t),t)dt \\ &= \left(f_x(x(t),t)x'(t) + f_t(x(t),t)\right) dt \end{aligned}$$

  by the ordinary rules of calculus.
- If $x(t)$ is replaced by Brownian motion, however, then a change of δt in t changes $f(x(t),t)$ by an amount proportional to $\sqrt{\delta t} f_x(x(t),t)$ (with a random coefficient).
- And the next term in the Taylor expansion with respect to x is proportional to δt. Not negligible.
- The idea is

$$f(X(t)+dX_t,t) = f(X(t),t) + f_x(X(t),t)dX_t + \frac{1}{2} f_{xx}(X(t),t)(dX_t)^2$$

- And the $(dX_t)^2$ term is like dt.

Ito's Formula

- There are various versions of this formula. First version:
- B_t is standard Brownian motion.
- $f(x,t)$ is twice differentiable in x and once in t.
- Then

$$df(B_t,t) = (f_t(B_t,t) + \frac{1}{2}f_{xx}(B_t,t))dt + f_x(B_t,t)dB_t$$

- This means

$$f(B_T,T) = f(0,0) + \int_0^T \left(f_t(B_t,t) + \frac{1}{2}f_{xx}(B_t,t)\right)dt + \int_0^T f_x(B_t,t)dB_t.$$

Example

- Take
$$f(x,t) = x_0 \exp\{(\alpha - \beta^2/2)t + \beta x\}$$
- Then
$$\begin{aligned} f_x(x,t) &= \beta f(x,t) \\ f_{xx}(x,t) &= \beta^2 f(x,t) \\ f_t(x,t) &= (\alpha - \beta^2/2) f(x,t) \end{aligned}$$
- So if $X_t = f(B_t, t)$ we have
$$dX_t = df(B_t,t) = \left((\alpha - \beta^2/2)X_t + \frac{1}{2}\beta^2 X_t\right) dt + \beta X_t dB_t.$$
- So Geometric Brownian motion solves
$$dX_t = \alpha X_t dt + \beta X_t dB_t$$

Ito's Formula Generalized

- Second version:
- B_t is standard Brownian motion.
- We have a process X which solves

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t$$

- $f(x, t)$ is twice differentiable in x and once in t.
- Then

$$\begin{aligned} df(X_t, t) = (f_t(X_t, t) + \mu(X_t, t)f_x(X_t, t) + \frac{\sigma^2(X_t, t)}{2} f_{xx}(X_t, t))dt \\ + \sigma(X_t, t)f_x(X_t, t)dB_t \end{aligned}$$

- Essentially we get $f_x(X_t, t)dX_t$ and $f_{xx}(X_t, t)(dX_t)^2/2$.
- Then we ignore terms like $dtdX_t$ and so on in squaring out.
- But we use $(dB_t)^2 = dt$.

Second Example

- For Ornstein Uhlenbeck put

$$U_t = \int_0^t e^{\alpha s} dB_s$$

so

$$dU_t = e^{\alpha t} dB_t$$

- Then define

$$V_t = e^{-\alpha t} \left\{ V_0 + \sigma \int_0^t e^{\alpha s} dB_s \right\}.$$

- Define

$$f(x,t) = e^{-\alpha t} \left\{ V_0 + \sigma x \right\}.$$

- So

$$V_t = f(U_t, t).$$

Example Continued

- We find

$$f_t(x,t) = -\alpha f(x,t)$$
$$f_x(x,t) = \sigma e^{\alpha t}$$
$$f_{xx}(x,t) = 0$$

- And that gives

$$dV_t = -\alpha V_t dt + \sigma dB_t.$$

- This is the Ornstein Uhlenbeck SDE as advertised.