STAT 890: Assignment 2

Instructions: In this assignment I want you to introduce to the tools of characteristic functions. You will need some tools which you can use without proof; others I will give hints to help you develop them.

Suppose that $X_n, n = 1, 2, \cdots,$ X and are real value random variables. Let ϕ_n, ϕ and ψ be the corresponding characteristic functions.

Theorem 1 (Stone-Weierstrass) *If K, d is a compact metric space and $\mathcal{A}$ is an algebra of real valued functions which separates points and contains the constant functions then $\mathcal{A}$ is dense in the space $\mathcal{C}(K)$ of all continuous real valued functions on K.*

Here are 3 such algebras:

$\mathcal{T}_1$: the collection of all polynomials on any interval K in the real line is such an algebra.

$\mathcal{T}_2$: if K is an interval in in $\mathbb{R}$ then the collection, $\mathcal{T}$, of all (finite) linear combinations of functions of the form $x \mapsto \cos(tx)$ or $x \mapsto \sin(tx)$ is such an algebra.

The third example is slightly more complicated to describe. Let K be the unit circle in the complex plane (or in $\mathbb{R}^2$). Consider the maps

$$g_k : z \mapsto \text{Re}(z^k)$$

and

$$h_k : z \mapsto \text{Im}(z^k)$$

Any product of linear combinations of these maps can be rewritten as a linear combinations of these maps so the family of all such linear combinations is an algebra $\mathcal{T}_3$ which separates points and vanishes nowhere. The identities

$$\begin{aligned} \text{Re}(z^k) &= \frac{z^k + \bar{z}^k}{2} \\ \text{Im}(z^k) &= \frac{z^k - \bar{z}^k}{2i} \\ \bar{z} &= 1/z \qquad \text{if } |z| = 1 \end{aligned}$$

can be used.

In the work which follows you need a trick. Suppose g is a continuous function with the property that $g(x) = 0$ for all x such that $|x| \geq M$ for some M. Let g_1 be g extended to be periodic with period $2M$. Let h be the function on C defined by

$$h(e^{it}) = g_1(x)$$

for any x with

$$\exp\{\pi i x/M\} = \exp(it)$$

There are many real t which give the same value of e^{it} and many real x which give the same value of $\exp\{\pi i x/M\}$ but since g_1 is periodic we see that h is uniquely defined. This function h is continuous on C. If we find a map h^* in the last algebra above which comes within ϵ of h on C then h^* defines a map g^* on the real line which is a linear combination of functions of the form

$$\exp\{k\pi i x/M\}$$

which is everywhere within ϵ of g_1. Notice that g^* is periodic with period $2M$.

Theorem 2 *If X and Y are two real random variables and*

$$E(g(X)) = E(g(Y))$$

for all bounded continuous functions g then X and Y have the same distribution

$$P^X = P^Y$$

1. Rather than proving the second theorem prove that under the conditions given X and Y have the same cdf.

2. Separates points means that for each pair x, y of points in K there is an element f of the algebra with $f(x) \neq f(y)$. Prove that $\mathcal{T}$ separates points.

3. Suppose g is a continuous function on $\mathbb{R}$ with compact support, that is, with the property that there is a compact set K such that $g(x) = 0$ for $x \notin K$. Use the Stone Weierstrass theorem and the second theorem above to prove that if $\psi(t) = \phi(t)$ for all t then

$$E(g(X)) = E(g(Y))$$

4. Use the dominated convergence theorem to prove that for integrable X we have

$$\lim_{n\to\infty} E\left\{X1(|X|>n)\right\} = 0$$

5. Now use the results so far to show that X and Y have the same distribution.

6. A sequence of random variables X_n is **tight** if, for every $\epsilon > 0$ there is a compact set K such that

$$P(X_n \notin K) \le \epsilon$$

for all n. Suppose now that $\phi_n(t)$ converges to some $\phi(t)$ in a neighbourhood of 0 and that ϕ is continuous at 0. Fill in the details in the following proof that the sequence X_n is tight:

Compute

$$\frac{1}{2u}\int_{-u}^{u} \phi_n(t)dt = \mathrm{E}\left[\frac{\sin uX_n}{uX_n}\right]$$

by justifying taking the integral inside an expected value.

Justify taking the limit of the left hand side as $n \to \infty$ for u sufficiently small and getting the same thing with ϕ_n replace by ϕ.

Explain why that limit is within ϵ of 1, when u is taken sufficiently small.

Argue that there is an $\alpha < 1$ such that

$$|y| > 1 \text{ implies} |sin(y)/y| < \alpha$$

Argue that the expected value on the right above is bounded by

$$P(|uX_n| < 1) + \alpha P(|uX_n| \ge 1)$$

Combine these to deduce that for some $u > 0$ and some N and all $n \ge N$ we must have

$$P(|uX_n| \ge 1) \le \epsilon$$

which is tightness.

7. Now suppose $\phi_n(t) \to \phi(t)$ for each t. Show that for each bounded continuous g we have

$$E(g(X_n)) \to E(g(X))$$

Hint: used the bound on g and tightness to find an interval $[-M, M]$ such that

$$\mathrm{E}\left[g(X_n)1(|X_n| > M)\right] < \epsilon$$

for all n and similarly for X. Approximate g on $[-M, M]$ uniformly by a member h of the algebra $\mathcal{T}_3$ above. Argue that

$$\mathrm{E}\left[h(X_n)1(|X_n| > M)\right] < \epsilon'$$

for some suitable ϵ'.

8. **This part is no longer relevant** Fix an $\epsilon > 0$ and a general bounded continuous function, g. Show, using the dominated convergence theorem, that there is an $M < \infty$ such that

$$E\left\{|g(X)1(|X| > M)\right\} < \epsilon.$$

Due: October 4, 2006.