STAT 890: Assignment 3

Instructions: In this assignment I want you to introduce to the tools of characteristic functions.

1. Prove that in any separable metric space if $X_n \to X$ in probability or almost surely then
$$X_n \Rightarrow X$$

2. Show that Fatou's lemma implies the Dominated Convergence theorem.

3. A family $\{X_\alpha; \alpha \in A\}$ of real random variables is **uniformly integrable** if
$$\lim_{M\to\infty} \mathrm{E}\left[|X_\alpha| 1(|X_\alpha| > M)\right] = 0$$
This problem concerns a triangular array weak law of large numbers. Suppose $\{X_{nj}; j = 1, \ldots, k_n; n = 1, 2, \cdots\}$ is a triangular array of mean 0 random variables and that for each fixed n the variables $X_{n1}, \cdots, X_{nk_n}$ are independent. Assume the array is uniformly integrable. Let $\{a_{nj}; j = 1, \ldots, n; n = 1, 2, \cdots\}$ be a triangular array of positive constants such that for each n we have
$$\sum_j a_{nj} = 1.$$
and assume that
$$\lim_{n\to\infty} \max\{a_{nj}; 1 \le j \le n\} = 0.$$
Show that
$$Y_n \equiv \sum_j a_{nj} X_{nj} \to 0$$
in probability as follows.

   (a) Fix any set of positive constants $\{b_{nj}, 1 \le j \le k_n, n = 1, 2, \cdots\}$. Let
   $$Y_n^* = \sum_j a_{nj} X_{nj} 1(|X_{nj}| \le b_{nj})$$
   Show that
   $$b_n \equiv \min_j\{b_{nj}\} \to \infty$$

implies

$$\mathrm{E}(S_n^*) \to 0$$

and

$$\mathrm{E}(|S_n - S_n^*|) \to 0$$

(b) Show that there is a choice of constants b_{nj} for which

$$\mathrm{Var}(S_n^*) \to 0$$

(c) Finish the problem.

(d) I believe, but have not checked, that the converse is also true: if the triangulare array has mean 0 and is independent within rows and is not uniformly integrable then there is a choice of constants $\{a_{nj}\}$ satisfying the given hypotheses but with

$$\sum_j a_{nj} X_{nj}$$

not converging to 0 in probability.

4. If a family $\{X_a; a \in A\}$ is uniformly integrable then

$$\mathrm{E}\left[e^{itX_\alpha}\right] = 1 + it\mathrm{E}(X_a) + R_a(t)$$

where the functions R_a are uniformly $o(t)$; that is

$$\sup\{|R_a(t)|; a \in A, |t| \le h\}/h \to 0$$

as $h \to 0$.

5. In the Cauchy model we looked at consider the following estimate of α: let $\tilde{\alpha}$ be the least value of α such that at least half of the data points satisfy $Y_i - \alpha x_i < 0$. Assume all $x_i > 0$ if that helps. Prove that $\tilde{\alpha}$ is consistent for α.

Hint: Let

$$G_n(\alpha) = \frac{1}{n} 1(Y_i - \alpha x_i < 0)$$

Prove that for $\alpha < \alpha_0$ the sequence $G_n(\alpha)$ has a limit which is strictly less than 1/2. For $\alpha > \alpha_0$ the limit is strictly greater than 1/2. Use this to prove that $\tilde{\alpha}$ is consistent. You may be able to use moments to show these limits exist.

6. In fact you may be able to do the following: find a sequence $\epsilon_n \to 0$ with

$$P(G_n(\alpha_0 - \epsilon_n) < 1/2 < G_n(\alpha_0 + \epsilon_n)) \to 1$$

This would prove

$$P(|\tilde{\alpha} - \alpha_0| \le \epsilon_n) \to 1.$$

(It is likely that you would need ϵ_n large compared to $n^{-1/2}$.)

Due: October 25, 2006.