Metric Spaces

Definition: A *metric space* is an ordered pair (S, d) where S is a set and d a function on $S \times S$ with the properties of a *metric*, namely:

1. $d(x, y) = d(y, x) \geq 0$.

2. $d(x, y) = 0$ iff $x = y$.

3. The triangle inequality holds:

$$d(x, z) \leq d(x, y) + d(y, z)$$

   for all x, y, z in S.

Definition: For $r > 0$ the set

$$B_r(x) = \{y \in S : d(x, y) < r\}$$

is the *open ball* of radius r centered at x.

Definition: A subset O of S is *open* if, for every $x \in O$ there is a $r > 0$ such that $B_r(x) \subset O$.

Notice that the empty set $\emptyset$ and S are open.

Definition: A subset C of S is *closed* if its complement C^c is open.

Notice that the empty set $\emptyset$ and S are closed.

Fact: an arbitrary union of open sets is open; an arbitrary intersection of closed sets is closed.

Definition: The closure in S of a set A (denoted $\overline{A}$) is the intersection of all closed subsets of S containing A.

Definition: A set D is *dense* in S if $\overline{D} = S$.

Definition: A metric space S, d is *separable* if it has a countable dense subset.

Limits

Definition: We say

$$\lim_{x \to y} f(x) = a$$

if, for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_1(x, y) \leq \delta \Rightarrow d_2(f(x), a) \leq \epsilon$$

Closed is equivalent to containing all its *limit points*.

Definition: We say x is a limit point of A if there is a sequence x_n of points in A converging to x.

Continuous Functions on Metric Spaces

In the following S_1, d_1 and S_2, d_2 are two metric spaces.

Definition: A function $f : S_1 \mapsto S_2$ is *continuous* if whenever O_2 is open in S_2 the inverse image

$$f^{-1}(O_2) = \{x \in S_1 : f(x) \in O_2\}$$

is open in S_1.

Theorem 1 *A map $f : S_1 \mapsto S_2$ is continuous iff*

$$\lim_{x \to y} f(x) = f(y)$$

for all $y \in S_1$.

Proof of Theorem: Suppose f is continuous. Fix y and $\epsilon > 0$. Take $O_2 = B_{f(y)}(\epsilon)$. Then $O_1 = f^{-1}(O_2)$ is an open set containing y. This means there is a $\delta > 0$ such that $d_1(x, y) \leq \delta$ implies $x \in O_1$ which means $f(x) \in O_2$.

Conversely suppose f satisfies the given condition. Suppose O_2 is open in S_2 and let $y \in f^{-1}(O_2)$. Let z denote $f(y)$ and note $z \in O_2$. As such there exists $\epsilon > 0$ such that $d_2(v, z) \leq \epsilon$ implies $v \in O_2$. Let δ_y be the corresponding δ from the condition.

Notice that if $x \in B_{\delta_y}(y)$ then $f(x) \in O_2$. That is $x \in f^{-1}(O_2)$. Thus

$$
\begin{aligned}
f^{-1}(O_2) &= \cup_{y \in f^{-1}(O_2)} \{y\} \\
&\subset \cup_{y \in f^{-1}(O_2)} B_{\delta_y}(y) \\
&\subset f^{-1}(O_2)
\end{aligned}
$$

Thus $f^{-1}(O_2) = \cup_{y \in f^{-1}(O_2)} B_{\delta_y}(y)$ is a union of open balls, so open.

Compactness

Definition: A family $\{O_\alpha; \alpha \in A\}$ of open subsets of S is an *open cover* of a set K if

$$K \subset \cup_\alpha O_\alpha$$

Definition: A subset K of S is compact if every open cover of K has a finite sub cover.

Every compact set is closed.

Example proof: Suppose $x \notin K$. For each y in K let $O_y = B_\epsilon(y)$ with $\epsilon = d(x,y)/2$. The family $O_y, y \in K$ is an open cover of K. Let $O_{y_1}, \ldots, O_{y_n}$ be a finite subcover. Let

$$\epsilon = \min\{d(x,y_j); 1 \leq j \leq n\}$$

and see that $B_\epsilon(x)$ has an empty intersection with each O_{y_i} by the triangle inequality. So $K \cap B_\epsilon = \emptyset$. The union of all these B_ϵ is the complement of K.

Definition: A subset K is totally bounded if, for each $\epsilon > 0$, K is contained in a finite union of balls of radius ϵ.

Theorem 2 *K is compact iff K is closed and totally bounded.*

Homework Problem: If f is a continuous map between S_1 and S_2 then the image $f(K)$ of a compact set K in S_1 is compact in S_2.

Definition: A norm, usually written $||\cdot||$ is a function on a vector space such that

1. For all x, $||x|| \geq 0$

2. $||x|| = 0$ iff $x = 0$.

3. $||ax|| = |a|\,||x||$ for each x and scalar a.

4. $||x+y|| \leq ||x|| + ||y||$.

Example: The set $S = \mathbb{R}^p$ with

$$d(x,y) = ||x-y||,$$

the usual Euclidean norm, is a separable metric space.

Every closed bounded set in $\mathbb{R}^p$ is compact.

Example: The set $\mathcal{C}$ consisting of all continuous functions from $[0,1]$ to $\mathbb{R}$ can be made into a metric space by taking

$$d(x, y) = \sup\{|x(t) - y(t)|; t \in [0,1]\}$$

$\mathcal{C}$ is separable.

Use Weierstrass theorem to show that the set of polynomials is dense in $\mathcal{C}$.

Then show that polynomials with rational coefficients are dense in the set of all polynomials.

Definition: A sequence x_n is **Cauchy** in S, d if: for each $\epsilon > 0$ there is an N such that $n \geq N$ and $m \geq N$ implies

$$d(x_n, x_m) \leq \epsilon$$

Definition: A metric space S, d is **complete** if every Cauchy sequence in S has a limit in S.

Fact: Ordinary Euclidean spaces $\mathbb{R}^p$ are complete.

Fact: $\mathcal{C}([0, 1]$ is complete for the uniform distance.