Probability Definitions

Probability Space : ordered triple $(\Omega, \mathcal{F}, P)$.

- Ω is a set (possible outcomes); elements are ω called elementary outcomes.

- $\mathcal{F}$ is a family of subsets (**events**) of Ω with the property that $\mathcal{F}$ is a σ-field (or σ-algebra):

  1. Empty set $\emptyset$ and Ω are members of $\mathcal{F}$.

  2. $A \in \mathcal{F}$ implies $A^c = \{\omega \in \Omega : \omega \not\in A\} \in \mathcal{F}$.

  3. $A_1, A_2, \cdots$ in $\mathcal{F}$ implies $A = \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

- P a function, domain $\mathcal{F}$, range a subset of $[0,1]$ satisfying:

  1. $P(\emptyset) = 0$ and $P(\Omega) = 1$.

  2. **Countable additivity**: $A_1, A_2, \cdots$ **pair-wise disjoint** ($j \neq k$ $A_j \cap A_k = \emptyset$)

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Axioms guarantee can compute probabilities by usual rules, including approximation.

Consequences of axioms:

$$A_i \in \mathcal{F}; i = 1, 2, \cdots \text{ implies } \cap_i A_i \in \mathcal{F}$$

$$A_1 \subset A_2 \subset \cdots \text{ implies } P(\cup A_i) = \lim_{n\to\infty} P(A_n)$$

$$A_1 \supset A_2 \supset \cdots \text{ implies } P(\cap A_i) = \lim_{n\to\infty} P(A_n)$$

Power set of any set A is a σ-field on A.

Intersection of arbitrary collection of σ-fields is σ-field.

In metric space S:

Borel σ-field in S: smallest σ-field in S containing every open ball.

Every common set is a Borel set, that is, in the Borel σ-field.

An S valued **random variable** is a map $X : \Omega \mapsto S$ such that when A is Borel then $\{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}$.

Fact: this is equivalent to

$$\{\omega \in \Omega : X(\omega) \in O\} \in \mathcal{F}$$

for all O open in S.

Jargon and notation:

Write $P(X \in A)$ for $P(\{\omega \in \Omega : X(\omega) \in A\})$

Definition: The **distribution** of X is the map

$$A \mapsto P(X \in A)$$

This is a probability on the set S with the Borel σ-field rather than the original Ω and $\mathcal{F}$.