The Forty-Ninth Annual William Lowell Putnam Competition
Saturday, December 3, 1988

Done

A–1 Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.

Solution: *If (x, y) is in R then so is $(\pm x, \pm y)$ for all choices of the signs. The desired area is thus 4 times the area of the part of R in the first quadrant. There the inequalities become*

$$0 \leq x \leq 1 + y \quad \textit{and} \quad 0 \leq y \leq 1$$

The area is

$$4 \int_0^1 \int_0^{1+y} dx dy = 4 \int_0^1 (1+y) dy = 4 \left. \frac{(1+y)^2}{2} \right|_0^1 = 6.$$

A–2 A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

Solution: *For this f we have*

$$(fg)' - f'g' = f'g + fg' - f'g' = (2xg(x) + g'(x) - 2xg'(x))f(x)$$

This is identically 0 if and only if

$$2xg(x) = (2x - 1)g'(x)$$

identically in some interval. For $g(x) > 0$ we would get

$$\frac{d \log g(x)}{dx} = \frac{2x}{2x-1} = 1 + \frac{1}{1-2x}.$$

Integrating gives $\log((g(x)) = x - 2\log(1 - 2x) + c$. *Take $c = 0$, $a = 0$ and $b = 1/2$. Then*

$$g(x) = \frac{e^x}{(1-2x)^2}.$$

Then compute the derivative to check.

A–3 Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1 \right)^x$$

converges.

Solution: *Since $n \sin(1/n) < 1$ for all n the terms in the sum are all positive. The sum therefore converges if and only if it converges absolutely. Put*

$$a_n = \sin(1/n) - 1/n + 1/(6n^3)$$

and

$$b_n = \frac{1}{n} \csc \frac{1}{n} - 1 = \frac{1}{n \sin(1/n)} - 1.$$

There is a constant C such that

$$|a_n| < C/n^5$$

for all n. Then

$$\frac{1}{n\sin(1/n)} - 1 = \frac{1/(6n^2) - na_n}{n\sin(1/n)}$$

As n tends to infinity the denominator converges to 1 so the sum in question converges if and only if

$$\sum \frac{1}{n^{2x}} \left(1 - n^3 a_n\right)^x < \infty.$$

For $x \le 0$ the quantities b_n^x do not converge to 0 so that the series diverges. Now consider only $x > 0$. The quantity $(1 - n^3 a_n)^x$ is bounded by a constant depending on x but not n and converges, as n goes to infinity to 1. Thus the sum converges if and only if $\sum n^{-2x}$ converges. This happens if and only if $x > 1/2$.

A–4 (a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?

(b) What if "three" is replaced by "nine"?

Solution: *If the answer to a) were no then the answer to b) would also have to be no.*

Call the colours R, G and B and let x be a point which is red. Then all points on the circle C_1 of radius 1 centered at x are either G or B. Consider the circle C_2 of radius $\sqrt{3}$ centred at x. I claim every point on C_2 must be coloured R. If y is such a point then the circle of radius 1 centred at y meets C_1 at two points, say, A and B. The distance between these two points is 1 because A, B and y form an equilateral triangle; that was how $\sqrt{3}$ was chosen. It follows that one of A and B is coloured G and the other is coloured B. Thus y is R and every point on C_2 is R. But there are other points on the circle C_2 which are at a distance 1 from y so the answer to (a) is Yes.

For **(b)***, consider a 3 by 3 array of squares of side r coloured with colours 1 to 9 in some order. For clarity include the bottom and left edge in the square but not the top or the right. If the first square has lower left corner at (0,0) then the points $(r, 0)$ and $(0, r)$ are not included in the square. Tile the plane with copies of these squares. If two points are in the same 3 by 3 square they are the same colour if and only iff they are in the same small square. The maximum distance apart for two such points is $\sqrt{2}r$. Again we agree that each subsquare includes the bottom and left side and only the lower left corner. If $r = 1/\sqrt{2}$ or less then there are not two points of the same colour in the same 3 by 3 square. If two points are in different squares, but the same subsquare (so as to have the same colour) then either their x co-ordinates or their y co-ordinates differ by at least $2r$. For $1/2 < r \le 1/\sqrt{2}$, then, there are no two points of the same colour exactly 1 unit apart.*

A–5 Prove that there exists a *unique* function f from the set R^+ of positive real numbers to R^+ such that

$$f(f(x)) = 6x - f(x)$$

and

$$f(x) > 0$$

for all $x > 0$.

Solution: *Begin by noting that $f(x) = 2x$ satisfies the identity.*

Now suppose f is any such function and x is some value for which $f(x) > 2x$. Put $x_0 = x$ and define inductively $x_{n+1} = f(x_n)$. Notice that

$$x_{n+2} = f(x_{n+1}) = f(f(x_n)) = 6x_n - x_{n+1}.$$

Since each $x_n > 0$ we see that $x_{n+2} < 6x_n$. This shows that

$$\phi(s) \equiv \sum_{n=0}^{\infty} x_n s^n < \infty$$

for all $0 \leq s < 1/6$. *Multiplying the identity above by* s^n *and summing from* $n = 0$ *to* ∞ *gives*

$$\frac{\phi(s) - sx_1 - x_0}{s^2} = 6\phi(s) - \frac{f(s) - x_0}{s}$$

which gives

$$\phi(s) = \frac{s(x_0 + x_1) + x_0}{(1 + 3s)(1 - 2s)}.$$

Write this as a partial fraction

$$\phi(s) = \frac{A}{1 + 3s} + \frac{B}{1 - 2s}$$

where $B = (3x_0 + x_1)/5$ *and* $A = (2x_0 - x_1)/5$. *By the formula for geometric series we see that*

$$x_n = A(-3)^n + B2^n$$

It follows that $x_n/(-3)^n$ *converges to* A *as* $n \to \infty$. *The denominator alternates in sign so unless* $A = 0$ *the numerator,* x_n *must eventually alternate in sign. Since each* $x_n > 0$ *we see* $A = 0$ *or* $x_1 = f(x_0) = 2x_0$. *Since this is true for all* x_0 *we see that* $f(x) \equiv 2x$.

A–6 If a linear transformation A on an n-dimensional vector space has $n + 1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

Solution: *Yes. Suppose* $(\lambda_1, v_1), \ldots, (\lambda_{n+1}, v_{n+1})$ *are eigenvalue/eigenvector pairs for* A *and that the* v_i *satisfy the condition given. If any* n *of the* λ_i *are equal, say the first* n *are all equal to* λ, *then I claim* A *is* λ *times the identity. To see this write any* v *as a linear combination of say* $v_1, \ldots, v_n$:

$$v = \sum c_i v_i.$$

Apply A *to see*

$$Av = \sum c_i A v_i = \sum c_i \lambda v_i = \lambda v$$

So $(A - \lambda I)v = 0$ *for every* v. *This proves* $A - \lambda I$ *is the 0 matrix.*
Let P *be the matrix whose columns are* $v_1, \ldots, v_n$ *and* Λ *be the diagonal matrix with entries* $\lambda_1, \ldots, \lambda_n$. *We have*

$$AP = P\Lambda \quad \textit{or} \quad A = P\Lambda P^{-1}.$$

Fix a column of P, *say column* i. *Let* P^* *be* P *with column* i *replaced by* v_{n+1} *and* Λ^* *be* Λ *with* λ_i *replaced by* λ_{n+1}. *We have*

$$AP^* = P^*\Lambda^*$$

The characteristic polynomial of A *is*

$$\det(A - \lambda I) = \det(P(\Lambda - \lambda I)P^{-1}) = \det(P)\det(\Lambda - \lambda I) det(P^{-1})$$

so that A *and* Λ *have the same characteristic polynomial. The argument applies also to* A *and* Λ^* *so that*

$$\det(\Lambda - \lambda I) \equiv \det(\Lambda^* - \lambda I).$$

These are diagonal matrices so

$$\prod_1^n (\lambda_j - \lambda) - \prod_{j \neq i} (\lambda_j - \lambda)(\lambda_{n+1} - \lambda) \equiv 0.$$

Thus

$$(\lambda_i - \lambda_{n+1}) \prod_{j \neq i} (\lambda_j - \lambda) \equiv 0.$$

Since the second term is not identically 0 we see $\lambda_{n+1} = \lambda_i$. *Since* i *was arbitrary we must have all the* λs *equal.*

B–1 A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2,3,4,\ldots\}$. Show that every composite is expressible as $xy+xz+yz+1$, with x,y,z positive integers.

Solution: *Write*

$$ab = (a-1+1)(b-1+1) = (a-1)(b-1)+(a-1)+(b-1)+1$$

Define $x=a-1$, $y=b-1$ and $z=1$ to finish the problem.

B–2 Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

Solution: *Let*

$$A = \{(x,y) : y \geq 0, y(y+1) \leq (x+1)^2\}$$

and

$$B + \{(x,y) : y \geq 0, y(y-1) \leq x^2\}.$$

The set A is the set of ordered pairs (x,y) with

$$0 \leq y \leq y_A(x) \equiv \frac{-1+\sqrt{1+4(x+1)^2}}{2}.$$

[The curve y_A is easily seen to be minimized at $x-1$ where $y_A(-1)=0$.] Similarly B is the set of ordered pairs (x,y) with

$$0 \leq y \leq y_B(x) \equiv \frac{1+\sqrt{1+4x^2}}{2}.$$

The curves y_A and y_B are continuous. If there is a point (x,y) in A but not B then we would have $y_A(x) > y_B(x)$. Since $y_A(-1) = 0 < (1+\sqrt(5))/2 = y_B(-1)$ there would have to be some x^* where the curves crossed, that is where $y^* = y_A(x^*) = y_B(x^*)$. But point (x,y) on the curve $y_A(x)$ satisfies $y(y+1) = (x+1)^2$ and each point on the curve $y_B(x)$ satisfies $y(y-1) = x^2$. The point (x^*, y^*) satisfies both these. But for any such point we may subtract the equations to find

$$2y = 2x+1 \quad \textit{or} \quad y = x+1/2.$$

Then

$$y(y-1) = (x+1/2)(x-1/2) = x^2 - 1/4 = x^2$$

which is a contradiction.

B–3 For every n in the set $\mathrm{N} = \{1,2,\ldots\}$ of positive integers, let r_n be the minimum value of $|c-d\sqrt{3}|$ for all nonnegative integers c and d with $c+d=n$. Find, with proof, the smallest positive real number g with $r_n \leq g$ for all $n \in \mathrm{N}$.

Solution: *Fix n and put $a_j = j-(n-j)\sqrt{3} = j(1+\sqrt{3})-n$. Since $a_0 < 0$ and $a_n > 0$ there is a j such that $a_j < 0 < a_{j+1}$. Since $a_{j+1}-a_j = 1+\sqrt{3}$ either the interval from 0 to a_{j+1} or the interval from a_j to 0 has length less than $(1+\sqrt{3})/2$. This shows $g \leq (1+\sqrt{3})/2$. I claim I can find a sequence of pairs of integers j_k, n_k with n_k tending to infinity and $j_k(1+\sqrt{3})-n_k \to -(1+\sqrt{3})/2$. For such a sequence we would have $(j_k+1)(1+\sqrt{3})-n_k \to (1+\sqrt{3})/2$ showing that $g \geq (1+\sqrt{3})/2$. It remains to produce the sequence of pairs j_k, n_k.*

Lemma: For any real x there is a sequence j'_k, n'_k of pairs of integers with

$$j'_k - x n'_k \to 0$$

as $k \to \infty$.

We will delay the proof of the lemma. Take $x = 1/(1+\sqrt{3})$. Find a sequence j'_k, n'_k as promised by the lemma. Then

$$2j'_k - 1 - 2n'_k x \to -1$$

Multiply by $(1+\sqrt{3})/2$. For $j_k = 2j'_k - 1$ and $n_k = 2n'_k$ we are done.

Proof of Lemma: *The claim is easy for rational x and easily reduced to the case $0 < x < 1$ which we now suppose. For each n find j_n such that $j_n - xn < 0 < j_n + 1 - xn$. Let $b_n = j_n - xn$. Since $-1 < b_n < 0$ for all n there is a b and an increasing sequence n_k with*

$$b_{n_k} \to b \in [-1, 0].$$

Notice that

$$b_{n_{k+1}} - b_{n_k} = (j_{k+1} - j_k) - x(n_{k+1} - n_k) \to 0.$$

From $n_{k+1} > n_k$ we deduce $j_{k+1} \geq j_k$ and so $(j_{n_{k+1}} - j_{n_k}, n_{k+1} - n_k)$ is the desired sequence of pairs. [Note: for irrational x it will be possible to pick a sequence of values of $n_{k+1} - n_k$ tending to infinity for otherwise x would have to be rational. I don't think I need this to get what I want, though.]

B–4 Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

Solution: *Let J be the set of integers n such that $a_n > 1/n^2$. For $n \in J$ we have*

$$\frac{(a_n)^{n/(n+1)}}{a_n} = \exp\{-\log(a_n)/(n+1)\} \leq \exp\{2\log(n)/(n+1)\}.$$

Let C denote the supremum over n of the right hand side. Note that $C < \infty$. Then

$$\sum_n (a_n)^{n/(n+1)} \leq \sum_{n \in J} (a_n)^{n/(n+1)} + \sum_{n \notin J} (a_n)^{n/(n+1)}.$$

The first term satisfies

$$\sum_{n \in J} (a_n)^{n/(n+1)} \leq \sum_{n \in J} C a_n < C \sum_n a_n < \infty.$$

The second term satisfies

$$\sum_{n \notin J} (a_n)^{n/(n+1)} \leq \sum_{n \notin J} \frac{1}{n^2} < \sum_n \frac{1}{n^2} < \infty.$$

So the indicated sum is finite.

B–5 For positive integers n, let M_n be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of M_n. (According to one definition, the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

One may note that

$$M_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

Solution: *Suppose $\mathbf{v}$ is a vector of length $2n+1$ such that $M_n\mathbf{v} = 0$. This gives a system of $2n+1$ equations. If we add equation i to equation $i+n$ we get the equation $v_i = v_{i+n}$ while if we add equation i to equation $i+n+1$ we get $v_i = v_{i+n+1}$. Thus*

$$v_{n+1} = v_1 = v_{n+2} = \cdots = v_{2n} = v_n = v_{2n+1}$$

so that $\mathbf{v} = c\mathbf{1}$ where $\mathbf{1}$ is a vector whose entries are all 1. Since $M_n\mathbf{1} = 0$ we see that the null space of M_n is 1 dimensional so that the rank of M_n is $2n$.

B–6 Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t, the number $at + b$ is a triangular number if and only if t is a triangular number. (The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, \ldots\}$.)

Solution: *Suppose $r \geq 0$ is an integer. Put $b = t_r$ and $a = (2r+1)^2$. Then*

$$\begin{aligned} at_n + b &= \frac{(2r+1)^2 n(n+1) + r(r+1)}{2} \\ &= \frac{\{(2r+1)n + r\}\{(2r+1)n + r + 1)\}}{2} \\ &= t_{(2r+1)n+r} \end{aligned}$$

for all $n = 0, 1, \cdots$. Conversely if

$$at + b = \frac{m(m+1)}{2}$$

for some m then we find the unique ℓ with

$$\ell(\ell+1) \leq 2t < (\ell+1)(\ell+2).$$

Then put $n = (2r+1)\ell + r$. We find

$$\begin{aligned} \frac{n(n+1)}{2} &= at_\ell + b \\ &\leq at + b = m(m+1)/2 \end{aligned}$$

The inequality is strict unless $t_\ell = t$. Moreover

$$\begin{aligned} \frac{(n+1)(n+2)}{2} &= \{n(n+1) + 2(n+1)\}/2 \\ &= at_\ell + b + n + 1 \\ &= at_{\ell+1} + b + n + 1 - a\ell \\ &= at_{\ell+1} + b + (2r+1)\ell + r + 1 - \ell \\ &= at_{\ell+1} + b + 2r\ell + r + 1 \\ &> at + b \\ &= \frac{m(m+1)}{2} \end{aligned}$$

Thus, unless $t = t_\ell$ we have

$$n(n+1) < m(m+1) < (n+1)(n+2)$$

which is impossible.