

Employment Uncertainty and Wage Contracts in Frictional Labor Markets *

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Abstract

Two essential aspects of many employment relationships are, (1) that they are meant to last a long time, and (2) that the participation of the worker and the firm in the match cannot be enforced during the relationship. I construct a model which takes the bilateral uncertainty of participation seriously, and characterize the optimal contract. In a frictional labor market, risk-neutral workers randomly receive opportunities to leave the firm. At the same time they are uncertain about their productivity in the match, and therefore about the (future) participation of the firm. Thus, in equilibrium, each party's participation decision depends on the expected participation decisions on the other side. In the model, firms post wage-tenure contracts, trading off instantaneous profit with the need to retain its workers, especially the workers in high-productivity matches. In the resulting wage-tenure schedule, wages can exhibit gradual growth over the course of the employment relationship, driven by (and responsive to) precisely the bilateral nature of uncertainty. I then solve for the equilibrium in the labor market. Uncertainty about the participation on both sides affects the extent of labor market competition, even though both parties are risk-neutral

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I Introduction

In the labor market, many workers and firms form relationships that are, in principle, meant to last a long time, and often they do¹. These relationships are not set in stone, however. In most cases either party can decide to leave the relationship at any time². For firms, the decision to leave a match (i.e. to dismiss the worker) could follow from an understanding that continued employment of the worker will no longer be profitable. Likewise, a worker could decide to leave a firm for alternative and better employment elsewhere, or to spend more time at home. Assuming that at least one of the parties likes the current employment relationship, i.e. it earns more in the match than it could by itself, it naturally worries about the participation of the other party.

In this paper, I study and characterize the labor contract offered by firms to workers, in a model with labor search frictions that takes seriously mutual uncertainty about the match. On the worker side, the uncertainty is his own productivity in the match - the worker may know, say, how hard he works, but does not know what value he yields to the firm and whether it is profitable to the firm to keep him. On the firm side, the uncertainty is the nature of outside offers that the worker may receive - the firm does not engage in outside offer matching (discussed below). At each point in time, each party's participation decision has to take into account the expected participation decisions on the other side. What kind of a contract would entice a worker to take a job whose tenure he cannot be certain of? How does the firm go about retaining good workers? How long should it keep the bad workers, if at all? What wages should it pay over time? These are the questions that I aim to answer in this paper, in a framework that incorporates the bilateral nature of employment uncertainty explicitly, in the setting of a frictional labor market³. (See the discussion of related literature in the next section). It is important that we understand how an employment relationship may be characterized in the presence of this bilateral uncertainty, as it is likely to affect not only the relationship itself, but, more broadly, the entire labor market and the dynamics of aggregate phenomena such as worker turnover and displacement.

To be concrete, the model incorporates the following key features. First, search frictions imply that a worker and firm can be better off together than each of them by themselves. Finding an alternative match is costly, and hence there is generally a surplus associated with being matched. Search frictions create an environment in which the desire for long-term relationships is natural on both sides of any match. Second, there is bilateral participation uncertainty during the relationship. On the one hand, workers search on the job and meet alternative employment opportunities randomly. On the other hand, the value of production is observable only to the firm, and not to the worker, which makes the worker uncertain about the firm's participation in the future. As

¹In 1996, 35.4% of workers between age 35-64 reported their job tenure to be greater than 10 years. 20.9% of workers between 45-64 years reported a tenure greater than 20 years. (Farber (1999), based on CPS data)

²This is a standard employment relationship in the U.S. (unless there is any evidence to the contrary, this is the default relationship), so-called "employment *at will*", where either party can terminate the relationship without notice for any reason, or for no reason at all. (Malcomson 1999, Malcomson 1997)

³Participation uncertainty matters in frictional labor markets. The frictions make it costly for the firm to find replacement workers. Therefore, worker turnover is often a major concern for firms. Moreover, employer to employer transitions occur frequently. Fallick and Fleischman (2004) document that 2.6% of workers move to another employer each month, and nearly 40% of the jobs started between 1994-2003 represent employer changes (based on CPS data). Likewise, from the worker side, the concern the worker often faces is the possibility of being let go, whether for external reasons (changes in the industry, say), or for reasons internal to his job that he may not be aware of (say, he cannot judge how valuable his work is for the firm). There are significant losses associated with displacement: dismissed employees often have to spend time in unemployment while looking for a new job, or have to be content taking a job that earns significantly less (Kletzer 1998).

the worker is not sure how valuable he is for the firm, he is uncertain how willing the firm is, at a point in time, to keep participating in the match.

Third, the firm commits to its wage contract over the employment relationship, conditional on the survival of the match. This is a stylized view, but one often taken in literature on frictional labor markets (see below). The contract specifies the amount that the firm pays the worker at each moment during the relationship, if both parties are still participating in the match. This wage schedule is posted and seen by every worker who meets the firm. Fourth, neither the worker nor the firm are obligated to stay in the match at any point in time. Finally, neither the worker nor the firm can respond when the other side decides to pull out of the match⁴. This inability to respond when the participation constraint becomes binding on the opposite side captures an empirically relevant feature of labor relationships, and places the model in a second-best world. Here, it is possible that parties walk away from relationships that are still creating surplus, and the constrained-optimal contract attempts to minimize such mistakes, while maximizing the overall payoff that falls to the firm -two conflicting motivations that have to be balanced out.

The firm aims to extract as much surplus as it can from each worker of any productivity. To keep the workers in the match, it wants -in general- to shift the date the compensation is paid out as much into the future as possible (this is known as backloading in the literature). At the same time, it values the participation of the *high productivity* worker more. Since the productivity of the match is unobservable to the worker, her participation depends, among other things, on the firm's expected participation for each productivity, and the firm takes this into account. So postponing compensation till the end may backfire as workers (in the bad, but also in the good matches) would be too uncertain about the participation of the firm, and they will be tempted to leave for other offers. The result is that the firm uses the wage schedule strategically, to reveal information to the worker about her productivity in order to retain her, but it does not want to do so too soon.

The major result of this paper is that the optimal contract exhibits a 'gradual' wage growth. Moreover, the amount of wage growth over the course of the relationship depends, in general, on the magnitude of the bilateral participation uncertainty faced by both parties. The model's features and outcomes can both speak to important observations in the data *and* they could tell us something new about these observations. For example, we do observe that wages generally increase over the tenure of a single job. At the same time we see turnover and dismissals decline over the employment relations, as in the model. On the other hand, we can gain new insights about the dependence of the contract on the characteristics and magnitude of the uncertainty that the parties face. This would allow us to link the contract to forces in the U.S. economy that affect the participation decisions on both sides, to study aggregate phenomena in the U.S. job market. I solve for the labor market equilibrium, in which each firm offers profit-maximizing wage-schedules, and where the high-value wage schedules provide the cause of participation risk of workers in matches are at lower values.

All in all, the major contribution of this paper is to propose an alternative theory of wage growth over the job tenure. To be sure, this is not the first paper to think about wage growth, but

⁴For the worker, if the firm decides to fire him, there is nothing he can do. For the firm, I assume that it does not engage in offer-matching. This assumption captures either the inability -perhaps caused by the difficulty of firms to see the outside offers of their workers- or the lack of desire of many employers in the data to engage in offer matching. Scott et al. (2004) find that nearly 80% of employers either did not make a counter-offer to any of their workers, or made a counter-offer to less than 5% of their workers, over the last year. (Their data is from the WorldatWork survey.) Weiss (1990), Mortensen (2003) and Manning (2005) discuss the reasons why firms prefer not to match.

it is a new approach that takes seriously crucial features of the labor market that have not been studied jointly before. I contrast the paper with alternative theories of wage growth in the next section.

The rest of the paper develops as follows. In the next section, I discuss the model in light of the related literature. Section 3 sets up the model, focusing on the decisions each party makes and defines the equilibrium. I derive the optimal wage-schedule in the section 4. In section 5, I solve for the equilibrium in the labor market, where firms choose which value the contract has that the post, and the turnover risk comes from from the initial values of contracts posted by other firms. In section 6, I discuss the result and its implications for more general settings. In particular, I discuss the potential impact of more general forms of uncertainty on the firm's side on wage contracts and wage growth. I briefly describe the implications of the model for aggregate phenomena and several extensions of this research agenda. Section 7 concludes.

2 Related Literature

The model fits in the strand of economics that models search frictions in the labor market explicitly. During the match, the worker can receive offers from other firms at random times, as in the on-the-job search literature (Burdett 1978 and thereafter). One of the standard ways in which the wage is formed in these models is posting: the firm is assumed to be able to commit to whatever wage or contract it posts. In equilibrium models with on-the-job search, there are two basic standard procedures for wage formation: wage bargaining (see, e.g., Pissarides 2000) and wage posting. I follow the latter route, because it allows me to formulate the problem of the optimal intertemporal division of payments over the relationship in a clean way. Moreover, I avoid the complications of modeling bargaining in the presence of asymmetric information. Burdett and Mortensen (1998) solve for the market equilibrium in the setting with wage posting and on-the-job search. However, they restrict the wages to be constant over the entire relationship. Shimer (1996) noted that in general, a constant wage is not the optimal policy in models with on-the-job search.

Stevens (2004) solves for the optimal wage-tenure contract (with time-varying wages) in a market where risk-neutral workers search on the job. Her result is that it is optimal for firms to engage in extreme backloading: the firm pays the minimum it is allowed to pay the worker for some time, and switches to paying the worker his productivity thereafter. Burdett and Coles (2003) solve a similar case, but, crucially, with risk-averse workers. Risk aversion drives their labor contract: wages increase gradually over the course of the employment relationship, but purely as a result of the desire to smooth consumption over time.

In neither Stevens, nor Burdett and Coles, is there any uncertainty about the firm's ability or willingness to pay up the large amounts it is promising later in the relationship. I, on the other hand, incorporate the notion that these promises are only as good as far as they are in the firm's interest to keep: a firm could promise to pay a lot at a much later date, and then simply fire the worker before the date comes. Moreover, the worker is uncertain about what exactly the firm's interest in the relationship is. In such an environment, it is much harder to sustain an employment relationship. One of the contributions of this paper is showing that uncertainty about the firm's participation can mean that extreme backloading is no longer optimal, even with *risk-neutral* workers and firms. To the best of my knowledge, (ex post) uncertainty about the quality of the match or firm, or about the firm's ability to pay wages in the future, has not been incorporated in models with on-the-job search (with the exception of Postel-Vinay and Turon

(2006), discussed below).

Two important elements of the model are that the firm cannot match outside offers and the firm cannot write a contract contingent on the match-specific productivity shock. This restricts the ability of the parties in the match to adjust to binding participation constraints on one of the sides. Following Burdett (1978), Mortensen (1989) and Burdett and Mortensen (1998), there is an entire strand of literature that takes the approach of not matching counteroffers (see Mortensen 2003 for an overview). If the firm and the worker could respond each time the participation constraint on the other side became binding, the future participation consideration would largely be absent in the determination of the current contract, as both parties would know that the contract can adjust to accommodate participation. An important dimension of the participation uncertainty would be eliminated. This is the case that Postel-Vinay and Turon (2006) study (with productivity shocks), following work of Postel-Vinay and Robin (2003, 2005) with heterogeneous firm productivities.

In contrast to Postel-Vinay and Turon, the model in this paper is set in a second-best world, where a worker who leaves the match does not take into account that it destroys the profit of the firm, and the firm does not take into account the value of employment that it destroys when firing the worker. The optimal contract in this paper trades off, ex ante, the participation risks after the match is formed. The major outcome is that wages rise over the course of the relationship, as an optimal response to these risks. Moreover, wages can rise relatively early, because the raise contains information (inferred in the equilibrium): if the match was bad, the firm would have fired the worker. It is the tension between backloading to prevent turnover and giving early raises to reveal information that provides the rationale for gradual wage growth on a job.

Of course, there are theories that can account for wage growth. One such theory is based on Jovanovic (1979). Here, workers and firms are uncertain about the quality of the match, and will gradually find this out over the duration of the relationship. Good matches will survive longest. The major difference between this theory and the present model is that it is not exogenous release of information that drives wage growth in the model in this paper, but rather the endogenous incentives of one informed party to tell the uninformed party about the quality of the match.

Perhaps the best-known theory of wage growth is the theory of human capital (Becker 1962). Not all observations on the labor market can be easily accounted for in the *pure* human capital story (in its basic forms). General human capital (transferable across employers) for example, is thought to explain wage growth with labor market experience. However, displaced workers often face prolonged losses of income. Firm-specific human capital can be an alternative explanation, but the mechanism by which skills that are not valued at other firms, and not priced by the market, are rewarded, is unclear. It is often imposed that there is a certain risk that outside offers of varying quality arrive during the relationship (see Manning's 2005 discussion) to force the firm to increase wages for a worker with firm-specific human capital at least partially. However, in a setting with on-the-job search, as illustrated in this paper, wage growth can occur without firm-specific human capital⁵. Of course, it is an interesting question to distinguish empirically between these causes of wage growth, one that has spawned a large literature, and no general agreement⁶. Thus, while this paper does not want to downplay the importance of human capital, it also holds that there is

⁵Moreover, a number of papers (Medoff and Abraham 1980, 1981, Kotlikoff and Gokhale 1992 and Abowd and Dygalo 2005) have found that productivity and wages do not rise in a proportional relation, instead the older workers are paid a higher fraction of their productivity, which appears to support that there are other motivations than productivity alone for the observed wage profiles

⁶One only needs to look at the discussion on the returns to tenure, for example.

room for proposing and investigating alternative theories.

3 Model

In the current model I do not take all causes of job insecurity head on; instead I opt for a stylized view, focussing on one aspect of the uncertainty that the worker faces about the firm. The worker cannot see the value of his production in the match, and this value is an important determinant in the participation decision for the firm. This allows us to capture a key trade-off between facilitating the worker's and the firm's participation, while at the same time it sets up the foundation for applying the model to other sources of job insecurity.

3.1 Setup

A measure ι of firms and an identical measure of workers live in continuous time. Workers and firms meet at rate λ . Before a meeting occurs, a firm posts a wage-tenure contract $w(t)$; a firm is committed to pay this wage as long as the worker-firm match lasts. However, either party may leave the match at any time. Workers keep receiving outside offers while being matched, at rate λ , and drawn from a distribution $F(V)$ which is bounded from above.

Directly after the match is formed, a productivity of the match is drawn from a distribution $H(p)$ with bounded support, and observed by the firm, not the worker. Throughout the paper I focus predominantly on the case with a countable number of productivities. Given the posted wage-contract, the productivity determines how large the amount is that the firm is residual claimant of. However, given a non-stationary wage policy, the firm might find itself in the position that a previously profitable relationship becomes unprofitable. In this case, the firm will walk away from the relationship, but only *at the moment* that it is better to do so⁷

Limited commitment to on-going participation has a bite for the following assumption: firms cannot observe the offers that workers get while they are employed at the firm. As noted before, I assume that the firm will not engage in counteroffers. Weiss (1990) and Mortensen (2003) give a list of rationales why firms are reluctant to engage in offer-matching: workers can decide on the intensity of search for outside offers, and the firm wants to discourage searching, the firm does not have any practical verification mechanism for unobservable outside offers, and there are problems associated with unequal wages for identical jobs (the high wage guy gets fired first; or behavior of other workers in joint production is affected). All in all, the restriction on offer matching can be seen as capturing a relevant dimension of labor relations in a significant part of the labor markets. As a result, workers might very well leave the firm when the firm is still making positive profits. In other words, the match break-up is not necessarily privately efficient between the worker and the firm.

Moreover, on the firm's side, when wages become too high, the firm will have to fire the worker in a low productivity match, even though this worker is better off in the relationship at the current wage than in unemployment. All in all, the firm's and worker's possibility to leave

⁷I restrict myself here to 'simple' wage-tenure contracts, i.e. for each tenure t there is a single wage specified. Moreover, I assume that commitment is complete, in the sense that the firm always has to pay this amount whenever the worker works for him. Alternatively, one can focus on renegotiation proof contracts as well. This issue will be addressed as part of the further research agenda. Suffice to say that in many firms the pay-structure is to standardized enough to make the 'simple wage-tenure contracts' relevant. If one wants to have a concrete example of contracts that look like the contracts covered in the paper, one can think of up-or-out contracts, for example.

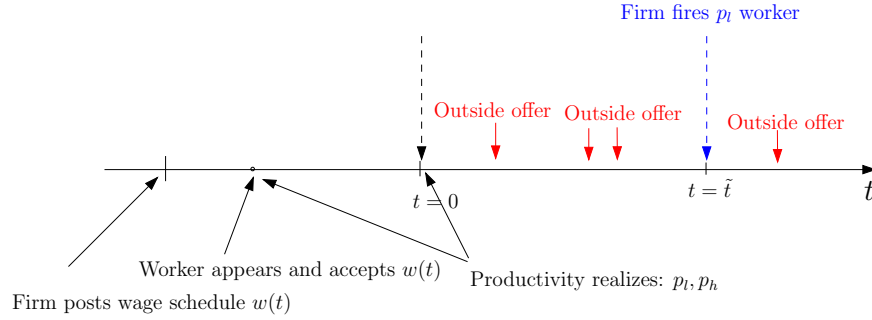


Figure 1: Timing

the other party behind while that party is still earning a surplus, create a central tension in the wage-schedule posted: too high wages will make the worker face considerable risk of being laid off, whereas low wages may lead the worker to quit his job quickly.

Workers and firms are both risk-neutral. However, I assume that the firm cannot sell the job in advance: at any time the wage has to be above some level c , which could be the minimum wage of a country, or the subsistence level of a credit-constrained worker⁸ Finally, I impose an exogenous *death* rate δ on the worker. Matches in the model are therefore terminated either by an exogenous death, after which the worker is replaced by an unemployed worker, or by the firm or worker. If the firm terminates the match, the worker becomes unemployed again.

To study a labor market with long-term relationships and bilateral participation uncertainty, there are two steps to be taken. First, and this is the focus of the paper at hand, I study the behavior within the match. In particular, I derive the optimal contract for the firm, taking the offers of other firms as given. Following this, the natural next step is to focus on the resulting market equilibrium, where each firm posts a contract of certain values to the worker, and acceptance probability of these contracts depends on the steady state distribution of workers over employment status and tenure.

3.2 Worker-Firm relationship

Let us concentrate on the relationship between a worker and a firm, who have just met, and where the firm previously has posted a wage-tenure contract with a life-time expected value V_0 . The value V_0 is an equilibrium object that summarizes the life-time utility that a worker can expect at $t = 0$ when he accepts the contract. It incorporates the payments of the firm to the worker, but also the outside offers that the worker finds profitable to take, and the unemployment benefit the worker gets in case he becomes unemployed.

The timing is as summarized in figure 1⁹. The firm's problem is to maximize expected profit, subject to the constraint of a 'simple wage-tenure contract' as defined above with should provide a

⁸I also rule out severance payments after firm-induced match-breakups, to make the worker-firm more resemble 'employment-at-will'. This would nevertheless be an interesting consideration, since it could distinguish the theory in the paper from other theories. A couple of authors have found that increases in job security (because of severance restrictions or payments) have led to more steeper wage-profiles. See e.g. Neumark and Stock (2001) and Osuna (2005), which is exactly what the model would predict. Note that models of on-the-job search with risk aversion make a similar assumption as the no-sale condition on the slope of the utility function, to rule out selling of the job.

⁹Note, that the restriction that the firm cannot see the productivity before the match is realized is not without loss of generality. It rules out that the firm uses the wage contract to signal the quality to the worker.

value V_0 to the worker at the beginning of the match. The firm knows that the wages it promises are discounted by the probability that the match will not survive up to the point. The total survival probability of the match depends as much on the expected probability that the worker has left the firm for a better opportunity elsewhere, as the expected probability that the firm will have found it better to end the match beforehand.

To be clear, when speaking of “ex ante” below, I am referring to the time of the contract posting, *before* the productivity has realized.

3.3 Decisions and Strategies

The **firm** takes two kinds of decisions. First, it decides to post a wage-tenure contract $w(t)$. Secondly, it has to decide at any point during the relationship whether to continue the match, or to walk away. The latter decision depends, of course, on the productivity p that has materialized, and the wage-tenure contract that the firm has posted before. It also depends on the firm’s expectations about the worker’s acceptance behavior. The first decision, which contract to post, depends on the firm’s ex ante expectation over the probability distribution of productivities that can realize at the beginning of the match. However, when posting the contract, the firm’s expectation about the worker’s behavior in accepting, and staying or leaving during the match given the posted contract, plays an equally important role.

The **worker** takes decisions about participation during and before the match. First, during a match, he constantly weighs whether he wants to continue or walk away. At a moment that he receives an outside offer, he compares the value he attaches to the outside offer, with the value he attaches to the continuation of the current relationship. To calculate the value of continuation, the worker needs to have expectations of the (probability of a) productivity in the match, and the future participation decisions of the firm. Furthermore, when the value of the match drops below the value of being unemployed, the worker will also leave the match. Lastly, before starting his employment, the worker bumped into the firm, made the same calculation whether to accept or reject the offer, but based on the value of in previous (un)employment.

Let $a_f(t)$ be the decision to continue in the match at time t for the firm, conditional on match survival up to that time. Then a strategy for the firm is the following:

- a wage-schedule $w(\tau) \in$ the space of functions with domain $[0, \infty]$
- a participation decision $a_f(t, p, w(\cdot)) \rightarrow [0, 1]$,

where one could in principle allow from continuation with some intermediate probability. Note that with $w(\cdot)$, I mean that the entire wage-tenure schedule is an argument in the strategy of the firm.

Let a_w denote a decision for the worker to continue with match. Then a strategy for the worker is a set of continuation decisions:

- an acceptance decision during the relationship when no outside offer is considered, i.e. whether to quit to unemployment or not: $a_w^u(t, w(\cdot)) \rightarrow [0, 1]$;
- an acceptance decision during the relationship when there is an outside offer of value \hat{V} is considered: $a_w^v(\hat{V}, w(\cdot), t) \rightarrow [0, 1]$;

When looking at the equilibrium in the entire market, there is a third acceptance function:

- an acceptance decision before the relationship, upon observing a wage schedule $w(\cdot)$, $a_w^x(w(\cdot)) \rightarrow [0, 1]$;

However, when focussing on the behavior within the match, I replace this dimension by the requirement that $w(\cdot)$ provides an expected value of V^0 or more to the worker.

As said, the value for the worker of staying in the relationship *evaluated at time t*, the *current* value $\tilde{V}(t)$, is an equilibrium object, given wage *schedule* $w(\cdot)$, and expectations about continuation actions on both sides. To keep notation simple, I will not index $\tilde{V}(t)$ by the equilibrium strategies. The tilde indicates that \tilde{V} is a current value, evaluated at time t. For simplicity let us assume that a worker will always break up the current match as long as the value of the outside offer is *weakly* higher than the value he attaches to staying in the relationship¹⁰. Optimal continuation decisions are therefore

$$a_w^v(\hat{V}, w(\cdot), t) = 0, \text{ iff } \tilde{V}(t) < \hat{V} \quad (1)$$

$$a_w^u(w(\cdot), t) = 0 \text{ iff } \tilde{V}(t) < V^u \quad (2)$$

With an arrival rate of λ of outside offers, the rate at which the worker leaves the match is then $\lambda(1 - F(\tilde{V}(t)))$ at each t . This implies the following survival function $\psi(t)$ for the continuation decisions of the workers., i.e. the probability that the match is still intact at time t , given $\tilde{V}(t)$ is given implicitly by

$$\dot{\psi}(t) = -[\lambda(1 - F(\tilde{V}(t)))]\psi(t), \quad (3)$$

which reduces to the following $\psi(t)$:

$$\psi(t) = e^{-\int_0^t [\lambda(1 - F(\tilde{V}(\tau)))]d\tau}, \quad \psi(0) = 1, \quad (4)$$

as long as $\tilde{V}(\tau) > V^u, \forall \tau \in [0, t]$, otherwise $\psi(t) = 0$.

Let us assume for now that each productivity will be broken at a unique time (or never at all),

$$\tilde{t}(p_i) = \arg \min_t a_f(t, p_i, w(\cdot)) = 0.$$

In any case, it is straightforward to show that whenever it is optimal to keep productivity p_i , it is optimal to continue matches with $p > p_i$. Hence, at any time the firm keeps all productivities higher than the lowest productivity in the match. Define therefore $\bar{p}(t)$ to be this lowest productivity still in the match. The survival function of the firm in the match is therefore at each t

$$[1 - H(\bar{p}(t))].$$

Now, everything is in place to look at value of being in the match, for both the worker and the firm.

3.4 Value Functions for Firm and Worker

One can define the value of a wage policy for the worker at time 0, and the current value $\tilde{V}(t)$, at each time t, given that the worker has (in equilibrium correct) expectations about ψ, \tilde{t}, \bar{p} , and takes $F(\hat{V}), V^u, H(p)$ as given. Let $h(p_i)$ be the probability that productivity p_i is drawn at the beginning of the match.

$$V(0) = \underbrace{\int_0^\infty e^{-\delta t} \psi(t) [1 - H(\bar{p}(t))] dt}_{\{1\}} \underbrace{\left(w + \lambda \int_{\tilde{V}(t)} \hat{V} dF(\hat{V}) \right)}_{\{2\}} + \underbrace{\sum_{p_i=\bar{p}} e^{-\delta \tilde{t}(p_i)} \psi(\tilde{t}(p_i)) h(p_i) V^u}_{\{3\}} \quad (5)$$

¹⁰From the analysis below it becomes clear that this assumption is not material.

To understand this value function, consider the term {1}: this is the probability that a match is still together at time t . Term {2} is the flow that the worker gets at each instant: he gets $w(t)$, and with probability λ he gets an outside offer. Upon accepting an outside offer, he gets the value of the outside offer, which I here, wlog, treat as if he consumes the outside offer instantly, in its entirety. In the next instant, $\psi(t)$ is decreased, because the match did not survive. The last term, {3}, incorporates probability of being dismissed and the resulting continuation value of unemployment at that time.

Along the same lines as V_0 one can derive current value $\tilde{V}(t)$. One has to correct for the fact that time index t is different, and that the remaining productivity distribution is different, if there were dismissals before t . Having said this, $\tilde{V}(t)$ is constructed exactly in the same way as $V(0)$ ¹¹:

$$\begin{aligned} \tilde{V}(t) = & \int_t^\infty \left(e^{-\delta(\tau-t)} e^{-\int_t^\tau \lambda(1-F(\tilde{V}(\zeta)))d\zeta} \underbrace{\frac{1-H(\tilde{p}(\tau))}{1-H(\tilde{p}(t))}}_{\psi(\tau)/\psi(t)} \left(w(\tau) + \lambda \int_{\tilde{V}(\tau)} \hat{V} dF(\hat{V}) \right) \right) d\tau \\ & + \sum_{p_i > \tilde{p}(t)}^{\bar{p}} e^{-\delta(\tilde{t}(p_i)-t)} e^{-\int_t^{\tilde{t}(p_i)} \lambda(1-F(\tilde{V}(\zeta)))d\zeta} \frac{h(p_i)}{1-H(\tilde{p}(t))} V^u, \end{aligned} \quad (6)$$

Note the following,

$$e^{-\int_t^\tau \lambda(1-F(\tilde{V}(\zeta)))d\zeta} = \psi(\tau)/\psi(t),$$

and naturally, $e^{-\delta(\tau-t)} = \frac{e^{-\delta\tau}}{e^{-\delta t}}$, so the following relation holds

$$V(t) = \psi(t)e^{-\delta t}[1-H(\tilde{p}(t))]\tilde{V}(t),$$

where $V(t)$ is the part of V^0 that realizes after time t . One can see that the value which a worker assigns to a wage schedule $w(\cdot)$ depends on the wages the firm promises to pay, the probability that these wages are actually paid (because the match might break up exogenous, by the firm or by the worker himself), the outside offers that the worker can take, and the probability and value of becoming unemployed. A higher wage schedule raises \tilde{V} and naturally increases the participation. However, this wage schedule is in a sense discounted by the firm's participation decisions. This can be seen clearly in the above equation, where the two factors in the value function that are directly determined by the firm, are

$$\frac{1-H(\tilde{p}(\tau))}{1-H(\tilde{p}(t))} \cdot w(\tau). \quad (7)$$

Thus, as the worker is more certain that he will get the wage payment (i.e. the first part of the term above is closer to one), a given wage-schedule will raise the current value more. Thus, the firm has two ways of creating a high \tilde{V} : higher wages, or a higher probability that they are paid.

Note that in the case of discrete productivities, the value for the worker who remains in the match will jump each time he expected the firm to fire a certain productivity. It tells him that he is of higher productivity, and that future rewards are now more likely.

Let us now focus on the profit for the firm. Given a productivity p , worker-survival function $\psi(t)$, wage schedule $w(\cdot)$, and $\tilde{t}(p)$ the conditional profit function is given by

$$\tilde{\Pi}(p, t) = \int_t^{\tilde{t}(p)} e^{-\delta(\tau-t)} \frac{\psi(\tau)}{\psi(t)} (p - w(\tau)) d\tau \quad (8)$$

¹¹Where I take the following stand on dismissals: if at time t a productivity is fired, I take $\tilde{V}(t)$ to be the current value after dismissals would have taken place, thus incorporated in \tilde{V} is that the worker *knows* it is not one of people being fired at time t .

The optimal continuation profit he can get from that moment onwards.

$$\tilde{\Pi}(p, t) = \max_{t'} \int_{\tilde{t}(p)}^{t'} e^{-\delta(\tau-t)} \frac{\psi(\tau)}{\psi(t)} (p - w(\tau)) d\tau \leq 0. \quad (9)$$

The dismissal time $\tilde{t}(p)$ has to satisfy the following: $\Pi(p, \tilde{t}) \leq 0$, and for no $\hat{t} < \tilde{t}(p)$ it is the case that $\Pi(p, \hat{t}) < 0$. Thus, if at any time t , the *current value* expected profit, i.e. the expected discounted profit at that time (conditional on surviving up to that point, and conditional on taking weakly optimal break-up decisions afterwards), is at zero, or strictly below zero, then the match can, respectively will be, terminated by the firm. From the equations above it follows immediately that $\tilde{t}(p)$ is indeed an *increasing* function of p . Likewise $\tilde{p}(t)$ is increasing.

Now, let us look at the ex ante stage, for the firm, where the firm evaluates which contract to post, ex ante profit for a wage schedule $w(\cdot)$, where I take $\psi(t)$, $\tilde{t}(p)$ or $\tilde{p}(t)$ from the appropriate subgame that follows the posting of a contract $w(\cdot)$. $\Pi^{\text{ex ante}}$ is then given by¹²:

$$\Pi^{\text{ex ante}} = \int_0^\infty e^{-\delta t} \psi(t) \left(\sum_{p_i=\tilde{p}(t)} (p - w(t)) h(p) \right) dt \quad (10)$$

Define, for future reference, the *ex ante expected* current value (ecv) of profit as well:

$$\Pi^{\text{ecv}}(t) = \int_t^\infty e^{-\delta\tau-t} \frac{\psi(\tau)}{\psi(t)} \left(\sum_{p_i=\tilde{p}(\tau)} (p - w(\tau)) h(p) \right) d\tau \quad (11)$$

3.5 Equilibrium Definition

Let us define the Within-Match Perfect Equilibrium

Definition 1 *The Within-Match Perfect Equilibrium is a set of functions $a_i(\cdot)$, $\psi(\cdot)$, $\tilde{p}(\cdot)$, $\tilde{t}(\cdot)$, $\tilde{V}(\cdot)$, $V(\cdot)$, $\Pi^{\text{ex ante}}$, $\tilde{\Pi}$, $w(t)$, $F(\hat{V})$, $H(p)$, and V^u such that*

1. **Second Stage:** Workers take optimal continuation decisions given \tilde{V} , V^u at any point during the relationship; ψ follows from these decisions of workers, through (3) and (4).
2. Firms take optimal continuation decisions, given $\tilde{\Pi}(t, p)$ as defined by (9), and the productivity p at any point t during the relationship. The breakup functions \tilde{p} , \tilde{t} , and survival function $(1 - G\tilde{p}(t))$ follow from the optimal continuation decisions of the firm.
3. ("Rational Expectations") The value functions \tilde{V} , V , $\Pi^{\text{ex ante}}$, P_i , $\tilde{\Pi}$ follow from ψ , \tilde{p} , \tilde{t} , $(1 - G\tilde{p}(t))$, $w(\cdot)$, given $F(\hat{V})$, $H(p)$, V^u through equations (5),(6),(9),(10),
4. **First Stage of the game:** $w(\cdot)$ is chosen, given $\psi(t)$, $\tilde{t}(p)$, V , \tilde{V} , Π , $\tilde{\Pi}$ that satisfy 1.-3. where the posted $w(\cdot)$ will be taken as given. Ex ante profit is constructed as in (10), and is maximized over the space of functions and subject to $V(0) \geq V^0$.

where $F(\hat{V})$, $H(p)$ and V^u are outside the match and taken exogenously.

¹²Or, alternatively by

$$\Pi^{\text{ex ante}} = \sum_{p_i=\underline{p}} \left(\int_0^{\tilde{t}(p_i)} e^{-\delta t} \psi(t) (p - w(t)) dt \right) h(p)$$

Note that the restriction to subgame perfect equilibrium implies the usual; concretely, I rule out equilibria where agents take non-optimal actions after some deviating wage contract $w_d(\cdot)$ is posted. I require that any $w(\cdot)$ is evaluated with functions $\psi(t), \tilde{t}(p)$, that are optimal in the subgame, i.e. satisfy conditions 1. through 3. This makes sure, e.g., that I don't have an equilibrium in which all workers in the economy will non-credibly threaten to leave the firm immediately, unless they get a contract that guarantee a certain value V .

3.6 Surplus of a Match

It is often easier to work with the joint surplus of the relationship, instead of looking separately at the value of the relationship to the firm $\Pi(t)$, and to the worker $V(t)$, or perhaps better, $V(t) - V^u$, the net value of being employed at the firm over being unemployed. Note that the assumptions on market and matching behavior imply that the firm can hire anybody who bumps into the firm. Thus, the outside option for the firm is zero. The value of unemployment is stationary, and can be derived as

$$\delta V^u = b + \lambda \int_{V^u} (\hat{V} - V^u) dF(\hat{V}).$$

Let J^{ecv} the expected current value of the surplus. It can then be shown that the value of the surplus is

$$\begin{aligned} J^{ecv}(t) &= \Pi^{ecv} + [1 - H(\tilde{p})(t)](V(t) - V^u) \\ &= \int_t^\infty e^{-\delta\tau} \sum_{p_i = \tilde{p}(\tau)} \psi(\tau) \left(p_i - b - \lambda \int_{V^u}^{\tilde{V}(\tau)} (\hat{V} - V^u) dF(\hat{V}) \right) h(p_i) d\tau, \end{aligned} \quad (12)$$

where the derivation of (12) can be found in the appendix. Note that, if one could hold the participation decisions of workers and firms constant, one could achieve any division of the current surplus: $w(t)$ does not turn up directly. However, $w(t)$ influences the surplus through the participation decisions. Thus at any moment t , future $w(\tau), \tau > t$ influences the participation of both firm and worker. The problem of the firm is to design the optimal contract that allows it to capture as much of the surplus as possible, taking the participation decisions into account.

4 The Optimal Wage Schedule

In this section, we concentrate once again on the worker-firm relationship. The issue at hand is to characterize the optimal wage schedule, given that it has to provide a certain minimum value to the worker. I approach the characterization problem in the following way. First, I show that an optimal response has certain properties, and then I limit the problem of finding optimal responses to the set of policies that has this property. After these reductions of the set of responses, I will solve for the equilibrium, within this set. The equilibrium outcome of this restricted problem is an equilibrium outcome in the original game as well.

The first result is an optimal contract always takes a particular form:

Proposition 1 *An optimal contract will have wages that are weakly increasing over time; the optimal quit decision therefore occurs at $w = p$. (Or, if wages jump up, at the first time that $w > p$.)*

Proof See the appendix. ■

Thus, even though there is a participation risk on both sides, the optimal contract has increasing wages. This is due to the different nature of the participation constraints on the two sides. The factor behind the firm's participation constraint is persistent, whereas the worker's outside options arrive at random moments *and* are randomly drawn each time. Thus, at each moment the worker might leave, and the fact that he has stayed up to now, does not say anything about tomorrow's chances of him leaving.

The random arrivals of outside offers, without persistence, makes the firm want to postpone wages¹³. The intuition can perhaps be illustrated as follows: suppose a firm has productivity 20. A worker works two periods for the firm. After the first period an outside offer arrives, which is 8 with probability 0.5, and 18 with probability 0.5. Suppose the firm originally gives an equal wage 10 in each period. The firm can do better by postponing. Note that originally with probability 0.5 the worker will leave the firm. Now keep the probabilities constant, and backload as much as necessary, keeping constant the total wages provided over the times that the worker *originally* wanted to stay. This means paying wage 18 in the last period, and 6 in the first period. Revealed preferences (1): The worker will (weakly) prefer this: he will get $6+18=24$ out of this relationship, the same as he got before when he would leave in 50% of the cases. The firm does strictly better. Revealed preferences (2): the firm pays the same amount over the realizations where the worker would stay originally, but now also has to pay when the worker chooses to stay where he did not before. But this is good for the firm: he now pays 18, keeps 2, where he would not have anything before. So, both worker and firm are better off, and the firm is strictly better off.

Offsetting the benefit of delayed wage increases is the fact that the firm might dismiss the worker, when the time comes for the wage raise. Dismissing the low-productivity worker could still be a policy that maximizes the (participation constrained) surplus, *ex ante*. In this case, the firm can promise more to the worker (in expectation), and therefore postpone more. On top of this, it could be possible (shown below in detail) that the firm wants to raise wages before, the inform the high productivity worker that he is in a good match, thus allowing the firm to postpone wages more for the high-productivity worker, after the firm's participation uncertainty is resolved or diminished. The latter is exactly the "conditional effect" in (7). The firm 'informs' the worker by raising wages to a level at which it is only profitable to keep the worker when the match is good. The persistence of the match productivity is important for the firm: it gives the firm both the motivation and possibility to raise the wages and make the argument to the worker in the high-productivity match. The remaining issue is what the best way is for the firm to do this. The crucial observation is that the worker knows that profitability of the low-productivity worker has fallen to or below zero. But this puts no direct requirements on *shape* of the future wage-schedule (i.e. whether it has to be upward or downward sloping, monotone, etc.); in fact, the firm still wants to backload as much as possible. What is the best way to tell the worker with minimum amount of frontloading: a wage that is raised, but then is as flat as possible, constant perhaps. Thus, the wage schedule will still be increasing.

Let us now proceed to characterize the contract more precisely. Since $\bar{p} < \infty$, I can restrict or focus on bounded wage-schedules as well. Since the optimal wage policy is monotone, it follows that $w(\cdot)$ is a.e. differentiable. Following the lead of Stevens (2004) one can construct right-differentiable (recursive) value functions between dismissal times, taking the values at the points of dismissal as boundary points.

¹³This motive appear e.g. in Harris and Holmstrom (1982), Burdett and Coles (2003), and Stevens (2004).

Consider the case of two productivities, p_l, p_h , where p_h arises with probability α .¹⁴ Time derivatives are denoted with a super-scripted dot, and the current value is denoted by a superscript cv -term. The expected current value, from an ex ante perspective (i.e. before the match productivity has realized) is indexed by ecv .

Firm's Profit Let us focus on the firm's profit first. The (*ex ante*) expected current value at time t , where $t \in [0, \tilde{t}(p_l))$ is given by

$$\dot{\Pi}^{ecv}(t) = (\delta + \lambda(1 - F(V^{cv}(t))))\Pi^{ecv}(t) - \alpha p_h - (1 - \alpha)p_l + w(t) \quad (13)$$

This function tells us, recursively, the expected current value of profit at each time. At dismissal time \tilde{t} (since there will be only one dismissal moment in the model, I simplify the notation), I define two values for profit: one in expected terms, before the uncertainty about the match quality is resolved, Π^- and one after the uncertainty is resolved Π^+ (and the match is p_h), as follows

$$\Pi^-(\tilde{t}) = \int_{\tilde{t}}^{\infty} (e^{-\delta(\tau-\tilde{t})} \frac{\psi(\tau)}{\psi(\tilde{t})}) (\alpha(p_h - w(\tau))) d\tau, \quad (14)$$

and

$$\Pi^+(\tilde{t}) = \int_{\tilde{t}}^{\infty} e^{-\delta(\tau-t)} \frac{\psi(\tau)}{\psi(\tilde{t})} (p - w(\tau)) d\tau; \quad (15)$$

it is clear that $\Pi^- = \alpha\Pi^+$. After dismissal, the then current value is given by:

$$\dot{\Pi}_h^{cv}(t) = (\delta + \lambda(1 - F(V^{cv}(t))))\Pi_h^{cv}(t) - p_h + w(t) \quad (16)$$

This can also be put in *ex ante expected terms*, by taking the time derivative of the ex ante profit function (following Stevens) between the dismissal times

$$\dot{\Pi}^{ecv} = \alpha\dot{\Pi}^{cv}(t) = (\delta + \lambda(1 - F(V^{cv}(t))))\alpha\Pi^{cv}(t) - \alpha(p_h - w(t)) \quad (17)$$

Perhaps the signs in the recursive formulation can come across as counter-intuitive. This is the formulation basically tells us how a current promise of, e.g. $\Pi_h^{cv}(t)$ is delivered (i.e. the promise at time t is held constant.): if the risk of match break-up today is $(\delta + \lambda(1 - F(V^{cv}(t))))$, where the firm will lose today's promise $\Pi_h^{cv}(t)$, and he gets a flow income of $p_h - w(t)$, it better be the case that the future promise makes up for any difference between the flow income, and the loss. Thus $\dot{\Pi}$ is increasing in the loss $(\delta + \lambda(1 - F(V^{cv}(t))))$, and decreasing in the flow income.

Worker's Value I can repeat the same exercise for the worker, using the following boundary conditions:

$$V^-(\tilde{t}) = \alpha \int_t^{\infty} e^{-\delta(\tau-t)} \frac{\psi(\tau)}{\psi(t)} \left(w(\tau) + \lambda \int_{\tilde{V}(t)} \hat{V} dF(\hat{V}) \right) d\tau + (1 - \alpha)V^u, \quad (18)$$

and, likewise

$$V^{cv}(\tilde{t}) = V^+(\tilde{t}) = \int_t^{\infty} e^{-\delta(\tau-t)} \frac{\psi(\tau)}{\psi(\tilde{t})} \left(w(\tau) + \lambda \int_{\tilde{V}(t)} \hat{V} dF(\hat{V}) \right) d\tau. \quad (19)$$

¹⁴The reasoning that I go through below applies just as well to any countable number of productivities, but the two productivity case is notationally simple, because there are only two phases: the interval before the worker in the low productivity match is fired, which coincides with the ex ante perspective, and the period after that worker is fired, in which all uncertainty on the firm's side is resolved.

Current value for workers inside both intervals:

$$\dot{V}^{cv}(t) = \delta V^{cv} - w(t) - \lambda \int_{V^{cv}} (\hat{V} - V^{cv}) dF(\hat{V}), \quad (20)$$

where before \tilde{t} , $V^{cv}(t) = V^{ecv}(t)$. Alternatively, I can write the latter equation as

$$\dot{V}^{cv}(t) = (\delta + \lambda(1 - F(V^{cv}))V^{cv} - w(t) - \lambda \int_{V^{cv}} \hat{V} dF(\hat{V}), \quad (21)$$

Surplus Between $t = 0$ and the time that the worker in the low-productivity match is fired, the expected surplus follows

$$j^{cv}(t) = (\delta + \lambda(1 - F(\tilde{V}))J^{cv} - (\alpha p_h + (1 - \alpha)p_l + b + \lambda \int_{V^u}^{\tilde{V}} (\hat{V} - V^u) dF(V) \quad (22)$$

I can rewrite this equation in simpler terms, without unemployment, and where I define $\Pi V \equiv \Pi^{ecv} + V^{ecv} = J^{ecv} - V^u$, for any t before dismissal time; and $\Pi V \equiv \Pi^{cv} + V^{cv} = J^{cv} - V^u$, for any t afterwards.

$$j^{cv}(t) = \delta \Pi V - (\alpha p_h + (1 - \alpha)p_l - \lambda \int_{V^{cv}} (\hat{V} - \Pi V) dF(\hat{V}) \quad (23)$$

From this formulation the private inefficiency is clear: the worker takes an outside offer whenever it is higher than is own value, but disregards the profit of the firm destroyed in the process. After low productivity worker is fired, the current value of the surplus becomes, conditional on a continuing match:

$$j^{cv}(t) = \delta \Pi V(t) - p_h - \lambda \int_{V^{cv}} (\hat{V} - \Pi V) dF(\hat{V}) \quad (24)$$

The boundary conditions, once again, are given by

$$J^- = \Pi^- + V^- - V^u,$$

and

$$J^+ = \Pi^+ + V^+ - V^u$$

Now, I have everything in place to study the shape of the contract offered in equilibrium. The first step is to show how the optimal contract behaves when there is no risk of dismissal (i.e. between two dismissal points), and no informational considerations; this problem was considered by Stevens (2004)¹⁵.

Lemma 1 *Given feasible boundary points V^0 at time 0, and V^- at some given equilibrium dismissal time \tilde{t} , a step-contract which has wages at either c or p_l is optimal for the firm. Likewise, for a contract after \tilde{t} with initial value V^+ , a step contract is optimal, with wages at either p_l or p_h .*

Proof Note that little freedom is given here: the time interval and the boundary values are fixed. From proposition 1, we know that wages will never decrease. Given that \tilde{t} was an equilibrium decision, it cannot be the case that it is better to lower below p_l during the second period, because that would yield a profitable deviation to delayed dismissal.

Let us concentrate on the first part of the lemma, the second part follows completely analogous. Index the step contract by s , the generic contract by g . The fact that the generic contract is

¹⁵The lemma below, though not identical, follows her proposition 5

feasible tells us that there is a wage schedule with $c \leq w \leq p_l$, for any t in our interval. Then, a constant wage of c over the interval would lead to $V(0) \leq V_0$, while a constant wage of p_l would lead to $V(0) \geq V_0$. By continuity of the value in the stepping time, there exists a step contract that is feasible (i.e. respects the boundary conditions). I want to show that it is contract that gives most to the firm. First, note that at any time $0 < t < \tilde{t}$, $V_s(t) \geq V_g(t)$ (strict if $w_g(s) \neq c, w_g(s) \neq p_l$ for some interval $[s_1, s_2]$: suppose not: $V_s(\hat{t}) < V_g(\hat{t})$, but then, if $w(\hat{t}) = c$, $V_s(0) < V_0$, because $\dot{V}_s \leq \dot{V}_g$ (strict if there is a strict difference in wages on some interval). Likewise, if $w(\hat{t}) = p_l$, then for $\hat{t} < t < \tilde{t}$, $\dot{V}_s \leq \dot{V}_g$ (going backwards from V^-), it must be that $V_s(\hat{t}) \geq V_g(\hat{t})$, strict if there is an interval of time in which wages differ. Hence $V_s > V_g \forall t$ in the interval. Now, look at J_s . Going backwards from \tilde{t} , both policies give a value V_0 to the worker. However, at any time $\hat{t} < \tilde{t}$, given that $V_s > V_g$ over the entire interval. Then, given that they end up at the same J^- , it must be that $J_s(0) > J_g(0)$. Since $\Pi_s(0) = J_s(0) - V_0 + V_u$, it follows that $\Pi_s(0) > \Pi_g(0)$. ■

This lemma shows that I only have to look at contracts that have wages at c , p_l or p_h . The next proposition tells me that I can restrict this even further: the firm will never offer p_l wages before it dismisses the low productivity worker, unless it offers wage p_l without dismissing at all.

First, notice the following about maximizing the expected surplus under static (time-invariant) wages. With time invariant wages the expected surplus is maximized by the highest wage possible, subject to the firm's participation constraint. I will index the values of such a policy with capital letters (L,H). Looking at equation (23), one can see that the surplus is diminishing in the distance between V and ΠV . A higher wage brings these closer together, naturally, because the worker will incorporate more of the value of the match when he gets an outside offer. For the case that firm does not fire the worker in a low-productivity match, the maximum static surplus has wage p_l and solves

$$\Pi V_L = \frac{(1 - \alpha)p_l + \alpha p_h + \lambda \int_{V_L} \max\{\hat{V}, \Pi V_L^*\} dF(\hat{V})}{\delta + \lambda}, \quad (25)$$

where from (20), it follows that

$$V_L = \frac{p_l + \lambda \int_{V_L} \max\{\hat{V}, V_L\} dF(\hat{V})}{\delta + \lambda}. \quad (26)$$

At this wage, firms make a profit over the workers in high-productivity matches:

$$\Pi_L = \frac{\alpha p_h}{\delta + (1 - F(V_L))},$$

and every time a high productivity worker gets an outside offer $\hat{V} > V_L$, he does not take Π_L into account.

On the other hand, I can look at the value under a $w = p_h$ policy. In this case, the firm will find it better to dismiss the p_l -worker. From (24), and looking in expected terms (in the equation below), one gets

$$\delta \Pi V_h = p_h + \lambda \int_{\Pi V_H^*} \left(\max\{\hat{V}, \Pi V_H\} - \Pi V_H \right) dF(\hat{V}) \quad (27)$$

$$\alpha \Pi V_H = \frac{\alpha p_h + \alpha \lambda \int_{\Pi V_H} (\max\{\hat{V}, \Pi V_H\}) dF(\hat{V})}{\lambda + \delta} \quad (28)$$

Note that for $J_H = \Pi V_H - V^u$, a firm with a low-productivity match will have dismissed its worker, whereas this worker still contributes to $J_L = \Pi V_L - V^u$. However, for the latter case matches with high productivity are mistakenly broken up, because the worker leaves, not taking into account the profit destroyed at the high-type firms. The trade-off therefore consists of match breakups due to shutting down low-productivity matches, and match breakups due to workers moving away from a highly-productive firm. This can be seen more accurately when rewriting the inequality for increasing efficiency with time-invariant wages, in ex ante terms

$$\alpha J_H > J_L.$$

Proposition 2 *In equilibrium, the firm breaks up a match with productivity p_i as soon as the wage w in the contract jumps up to p_i , except for the the highest productivity at the last jump.*

Proof See appendix ■

The intuition here is that when it is worthwhile to dismiss the worker in the low-productivity match, paying a wage p_l before dismissal effectively constitutes front-loading. The firm would be better off promising the surplus that falls to the worker after dismissal at a sooner time. If, on the other hand, wages at p_l would not constitute front-loading, this would contradict the optimality of the dismissal decision.

Now, let us look at the conditions under which the firm will offer p_h , after some (still unspecified) time \hat{t} . I find the conditions under which the firm prefers a jump straight to p_h , as opposed to jumping to p_l without dismissals, and show that if the firm would prefer to p_l *with dismissals* as a first step, it still prefers a big step to p_h to a step to p_l without dismissals.

Proposition 3 *The firms will offer a contract with p_h from some time \hat{t} onwards if and only if the maximum joint surplus of offering p_h is larger than the maximum joint surplus of offering p_l . If, moreover, the expected surplus αJ_H is large enough, it will offer p_l wages (after a possible dismissal) for some interval of time, before jumping to p_h .*

Firms will raise the wages higher than p_l if it is the expected joint surplus at constant wage p_h , αJ_H , is higher than the surplus at constant wage p_l , J_L . Moreover, when $\alpha J_H < J_L$, but the difference is sufficiently small, the firm will still find it optimal to raise wages above p_l ¹⁶. Note that latter implies that the joint surplus maximizing policy is not necessarily a constant wage policy! However, even with these non-constant wage policies, the jump still occurs if and only if it is (constrained) efficient that workers in low productivity matches are fired. By constrained-efficient I mean, in this context, privately efficient subject to the limited commitment of participation. In other words, this takes into account that some surplus cannot be reached, because one of the parties would walk away from the match.

Proof I show that a contract with all wages $w \leq p_l$ is dominated by some contract with some wages at p_h , if and only if the joint surplus of a contract with p_h is larger than a contract with p_l . From the previous analysis, we know that contracts without dismissals can only jump from c to p_l . Now, take a step contract with dismissals which, moreover, jumps straight to p_h at time \hat{t} . Repeating the argumentation of lemma 1, while taking into account that some matches will be broken up, it can be showed that this contract dominates any contract without dismissal, if

¹⁶The latter part looks like something that would happen with discrete productivities, but would disappear when one goes to a continuum of productivities.

and only if the conditions are satisfied. Note that at the time of the step, the (ex ante) value of the contract for the worker is $\alpha J_H + V^u > J_L + V^u$. For the firm, the contract that steps up to p_h at \tilde{t} has $\Pi(\tilde{t}) = \Pi_H(\tilde{t}) = 0$. I need to show that the ex ante gain for firms from posting a p_h -contract is larger, if and only if the conditions are satisfied.

To proceed towards this (and beyond), I have to introduce the recursive value functions for the surplus of three policies: J_l, J_d, J_h , where the lower-case letters denote that the contracts involve wages different than the productivity of the match. J_l tracks the *ex ante expected* surplus of a policy that has c up to some \tilde{t} , and then jumps up to p_l without dismissal. J_d follows the *ex ante expected* surplus of a policy that has c till \tilde{t} and then jumps up straight to J_H , with probability α . Finally, J_h follows the surplus, *after dismissal*, of a policy where the wage equals p_l till some \tilde{t}_2 , at which point it jumps up to p_h . I use *ex ante expected surplus* here because the firm is ultimately interested in the ex ante expected profit at time $t = 0$. \dot{J}_l and \dot{J}_d are used to trace back the expected surplus from J_L or αJ_H , to $J_i(0)$.

J_l is defined in the following way, taking $\mathbb{E}[p] = \alpha p_h + (1 - \alpha)p_l$:

$$\dot{J}_l \equiv \delta \Pi V_l - \mathbb{E}[p] - \lambda \int_{V(t)} (\hat{V} - \Pi V_l) dF(\hat{V}),$$

where $V \leq V_L$ at all times. b) the behavior of the surplus function of contract that jumps up at \tilde{t} to p_h , thus involving dismissal, and providing identical value $V(0)$ to the worker. The joint surplus evolution before the jump is dictated by J_d , with a d because workers still have to be dismissed¹⁷.

$$\dot{J}_d = \delta \Pi V_d - \mathbb{E}[p] - \lambda \int_{V(t)} (\hat{V} - \Pi V_d) dF(\hat{V}),$$

where V can increase up to $\alpha \Pi V_H + (1 - \alpha)V^u$. And, finally, after the jump surplus is given by

$$\dot{J}_h = \delta \Pi V_h - p_i - \lambda \int_{\Pi V_h} (\hat{V} - \Pi V_h) dF(\hat{V}),$$

with p_i either at p_l or p_h . There is the familiar before-and-after condition

$$J_d(\tilde{t}) = J^- = \alpha J_H$$

Now, let's trace back the joint surplus over time.

Lemma 2 *If $J_d > J_l$ for some τ , than $J_d > J_l$ for all $t \leq \tau$.*

Proof We are concerned with the case that V^0 is the same for both policies. Note that both policies dictate $w = c$ for some interval. Note that $V_l(t) \leq V_d(t)$ for all on this interval. Then suppose that $J_d(\tau) > J_l(\tau)$.

$$\dot{J}_d(\tau) = (\delta + \lambda(1 - F(V_d)))J_d - \mathbb{E}[p] - \lambda \int_{V_d} \hat{V} dF(\hat{V}) \quad (29)$$

$$< (\delta + \lambda(1 - F(V_l)))J_l - \mathbb{E}[p] - \lambda \int_{V_l} \hat{V} dF(\hat{V}) = \dot{J}_l(\tau); \quad (30)$$

Suppose that at some τ' $J_d = J_l$. Then we need to have $\dot{J}_d > \dot{J}_l$ at some interval $\tau' < t < \tau$, while $J_d(t) = J_l(t)$. By the equation (30) about this is not possible. Contradiction. Since I have

¹⁷Our previous notation J^{eccv} already captured this, but because I want to distinguish three contracts: p_l -step, no dismissal contracts, indexed by l , p_h -step dismissal contracts, indexed ex ante by d , and after the dismissal by h .

assumed the same initial condition V_0 , and I can alternatively define J_d as a function of V , not of t , where $J(V(t)) = J(t)$, over the time-interval where $w = c$. ■

Thus, if $J_d > J_l$ for any t , $V(t)$, then the dominating strategy for the firm is to offer a step contract with a step to p_h and a breakup of the low productivity matches.

Now, if $\alpha J_H > J_L$, then $\alpha J_H = J_d(V^u + \alpha J_H) > J_L(V_L)$. Thus, in this case, the dismissal decision is straightforward. However, also if $\alpha J_H \leq J_L$ (but not too low), it might be that the optimal policy asks for a dismissal.

Lemma 3 *If $J_d(V^u + \alpha J_H) = \alpha J_H$ is close to J_L , than $\dot{J}_d < 0$.*

Proof Subtract $\dot{J}_l = 0(V_L)$ from $\dot{J}_d(\alpha J_H + V^u)$, to find $\dot{J}_l < 0$

$$\begin{aligned} \dot{J}_l &= \dot{J}_d - \dot{J}_l \\ &\approx \delta \Pi V_d - \mathbb{E}[p] - \lambda \int_{\Pi V_d} (\hat{V} - \Pi V_d) dF(\hat{V}) - \delta \Pi V_l + \mathbb{E}[p] + \lambda \int_{V_L} (\hat{V} - \Pi V_d) dF(\hat{V}) \\ &= \int_{V_L}^{\Pi V_d} (\hat{V} - \Pi V_d) dF(\hat{V}) < 0, \end{aligned} \quad (31)$$

where $\Pi V_d = V^u + \alpha J_H$. As $J_d(\hat{t})$ increases, moving further away from J_l , $\dot{J}_d(\hat{t})$ will increase, as $d\dot{J}_d/dJ_d = \delta + \lambda(1 - F(J_d)) > 0$. ■

Now, I want to establish when there is an interval of time that wage p_l is offered. Let us first establish the slopes of the surplus, just before p_h will be given to the worker, for as long as he chosen to remain at the current match.

Lemma 4 *For V in a neighborhood of $V_H = J_H$, the optimal policy is wage p_l which jumps up to p_h , rather than a wage c which jumps up to p_h if and only if $J_d^{ss}(\alpha J_H) < \alpha J_H$.*

Proof Define

$$0 = j_d^{ss} = \delta \Pi V_d^{ss} - \mathbb{E}[p] - \int_{\Pi V_d^{ss}} (\hat{V} - \Pi V_d^{ss}) dF(\hat{V})$$

The first step is to prove that $J_d^{ss}(\alpha \Pi V_H) < \alpha J_H$, if and only if

$$\dot{J}_d(\alpha \Pi V_H) > 0,$$

where $\Pi V_H = \alpha J_H + V^u$. Take $\dot{J}_d(\alpha \Pi V_H)$ and subtract $\dot{J}_d^{ss} = 0$ from it

$$\begin{aligned} \dot{J}(\alpha \Pi V_H) &= \dot{J}_d(\alpha \Pi V_H) - \dot{J}_d^{ss} \\ &= \delta \alpha \Pi V_H - \mathbb{E}[p] - \lambda \int_{\alpha \Pi V_H} (\hat{V} - \alpha \Pi V_H) dF(\hat{V}) \\ &\quad - \delta \Pi V_d^{ss} + \mathbb{E}[p] + \lambda \int_{\Pi V_d^{ss}} (\hat{V} - \Pi V_d^{ss}) dF(\hat{V}) \\ &= \delta (\alpha \Pi V_H - \Pi V_d^{ss}) - \lambda \int_{\alpha \Pi V_H} (\hat{V} - \alpha \Pi V_H) dF(\hat{V}) \\ &\quad + \lambda \int_{\Pi V_d^{ss}} (\hat{V} - \Pi V_d^{ss}) dF(\hat{V}) \\ &> 0, \text{ if and only if } \Pi V_d^{ss} < \alpha \Pi V_H + (1 - \alpha) V^u \iff J_d^{ss} < \alpha J_H. \end{aligned}$$

The intuition is clear here: if one is at a lower steady state (J_d^{ss}), and now suddenly a switch is instituted to the high steady state αJ_H at some future time, the value of J will start to rise over

time, i.e. $\dot{J}_d > 0$. Likewise, if the value is rising while before we were at steady state value J_d^{ss} , we must be moving a new J , higher than steady state J_d^{ss} .

The second and last element that is needed is that $\dot{J}_h = 0, \dot{J}_d > 0$ implies that at least for a neighborhood of J_H , it is better to provide value through p_l , than through wage c .

To show this, note that given \dot{J}_h, \dot{V}_h , and terminal condition $J_H = V_H$, one can implicitly define a function $J_h(V)$. Likewise, an implicit function $J_d(V) = J_d(\alpha V + V^u)$ can be found, from \dot{J}_d, \dot{V}_h . Now,

$$\frac{d^-(\alpha J_h(V_H))}{d(\alpha V)} < \frac{d^-(J_d(V_H))}{dV},$$

where the minus-sign denotes the left-derivative at J_H and $(\alpha J_H + V^u)$, implies that given a V in some neighborhood close enough to $V = V_H$, it is better to choose the J_h policy (with wages p_l) than the J_d (with wages c , and dismissals still to come). But one can find

$$d^-(J_h(V_H))/dV = \frac{d^-(\alpha J_h)/dt}{d(\alpha V)/dt} = \frac{\alpha \dot{J}_h}{\alpha \dot{V}_h} = 0,$$

and

$$\frac{d^-(J_d(V_H))}{dV} = \frac{d^-J_d/dt}{d^-V/dt} = \frac{\dot{J}_d}{\dot{V}_d} > 0,$$

if and only if $J_d^{ss} < \alpha J_H$. Thus for some neighborhood of V_H it is optimal to have wages at p_l after a dismissal, if and only if this condition is satisfied.

Note that for the specification above a jump to p_l and then to p_h is also constrained efficient. Note that $J_L < J_d^{ss}(\alpha J_H) < \alpha J_H$, so the case directly above is a subset of the cases where constraint efficiency requires jumps to p_h , as derived before. It is just that p_h is preceded by some period of p_l . ■

Thus, above I have derived the condition for a wage schedule with wages at p_h at some time interval. It turns out that this jump will happen if and only if it is constrained privately-efficient to have a wage at p_h . The joint surplus-maximizing wage policy might call for a step contract itself (it is not necessarily a constant wage at one of the productivities, as one might have expected). The intuition for the latter contract is the following: the low productivity matches add a lot to the surplus, but the threat of outside offers that destroys the firm's profit is also large. Under the p_l regime, the firm cannot promise more than V_L , but under the J_d regime, the firm can. In this case, the ex ante expected wage surplus J_H is less than J_l , but the additional profit cashed in while V went from V_L to αV_H just compensates that. The second implication is that there might be some $\alpha J_H < J_L$ that still call for a dismissal as the joint surplus maximizing policy.

All in all, I have found the following: if $\alpha J_H > J_d^{ss}$, the contract has a step at p_l , before it jumps up to j_h . If $J_L < \alpha J_H < J_d^{ss}$, and even for some values below V_L , the optimal wage policy comes with one big step from c to p_h . Finally, if $\alpha J_H \ll J_L$, the optimal policy is a wage at c , followed by a raise to p_l , without dismissals. ■

Remark 1 *The case with the declining J_d function seems specific to the setup with discrete productivities.*

Note that in terms of efficiency and dismissals, I get the result that firms will dismiss workers if and only if it is ex ante constrained efficient to do so. Although, the decision whether to eventually dismiss workers or not coincides with constrained efficiency, the firms delay the decision to extract rents, so the timing of the dismissal is generally not ex ante constrained efficient.

Up to now, it was derived that the optimal contract is a step contract where the steps coincide with the productivities. The dismissals occur at the first step (if there is a second step, or a big jump straight to p_h). The crux of this is that the optimal contract can be characterized as a simple sequence of stepping times $\{\tilde{t}_i\}$, where $\tilde{t}_i = \tilde{t}_{i+1}$ is denoting a step of two productivities at once.

Thus, given the conditions are met for ending up at p_h , and having an intermediate step at p_l , the characterization of the stepping times comes from solving the following problem.

★ **The stepping time problem:**

$$\max_{\tilde{t}_1, \tilde{t}_2} \left(\int_0^{\tilde{t}_1} e^{-\delta t} \psi(t) (\alpha p_h + (1 - \alpha) p_l - c) dt + \int_{\tilde{t}_1}^{\tilde{t}_2} e^{-\delta t} \psi(t) (\alpha p_h - \alpha p_l) dt \right) \quad (32)$$

subject to

$$V_0 = \int_0^{\tilde{t}_1} e^{-\delta t} \psi(t) \left(c + \int_{\tilde{V}(t)} \hat{V} dF(\hat{V}) \right) dt + \alpha \int_{\tilde{t}_1}^{\tilde{t}_2} e^{-\delta t} \psi(t) \left(p_l + \int_{\tilde{V}(t)} \hat{V} dF(\hat{V}) \right) dt \\ + \alpha e^{-\delta \tilde{t}_2} \psi(\tilde{t}_2) V_H + (1 - \alpha) e^{-\delta \tilde{t}_1} \psi(\tilde{t}_1) V_H$$

and $\tilde{V}, \psi(t)$, derived analogous to (6), (4).

From this problem, the characterization of the stepping times, (given $\alpha J_H > J_a^{SS}$), is given by the following proposition.

Proposition 4 (a) *The conditions for the steps in a two-step contract are given by the following: going backwards in time, there is a step down from p_l to c , at time $\tilde{t}_2 - \tilde{t}_1$ when*

$$\frac{\dot{\Pi}^{ecv}(\alpha \Pi_h, \alpha V_h + (1 - \alpha) V^u)}{\dot{V}^{ecv}(\alpha V_h + (1 - \alpha) V^u)} = \frac{\dot{\Pi}^{cv}(V_h, \Pi_h)}{V^{cv}(V_h)} \quad (33)$$

at some Π, V , taking J, \dot{V} backwards from $V_H = J_H$.

Proof See appendix A for a short direct proof, and appendix B for a derivation of this result from setting up this problem as a so-called hybrid optimal control problem, and subsequently applying a generalized version of the maximum principle. ■

Whereas the proposition above finds the conditions that the stepping times have to satisfy, I am also interested in whether such a ‘first’ step to p_l occurs. One could think of cases in which it could be better to start offering wage p_l and firing the worker with a bad realization right away. A sufficient condition for this not to be the case (i.e. the wage contract will start at c), is given by the following:

$$0 = (\delta + \lambda(1 - F(V)))\alpha \Pi - \alpha(p_h - p_l) > (\delta + \lambda(1 - F(\alpha V)))\alpha \Pi - \alpha(p_h - p_l) - p_l + c$$

which leads to

$$\frac{\lambda(F(V) - F(\alpha V))}{\delta + \lambda(1 - F(V))} < \frac{p_l - c}{\alpha(p_h - p_l)}, \quad (34)$$

where I have used that $\dot{V} > 0$ in both cases. This condition basically tells us that the $\dot{\Pi}/\dot{V}$ lines will cross at some V , thus cause a step down, from p_l to c (if one were going backward). This sufficient condition is derived from the fact that $\dot{\Pi}^{ecv}(V) > 0$, while $\dot{\Pi}^{cv}(V) = 0$, for the

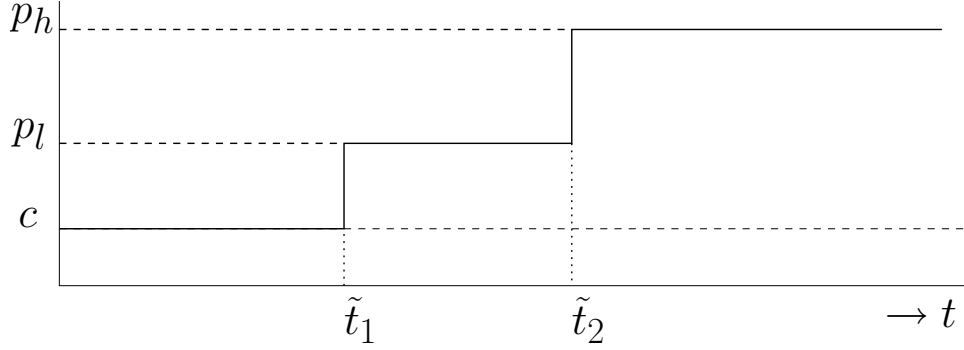


Figure 2: An optimal two-step contract

particular V in question. Then the step up will take place after that V , while it is at the same time profitable to offer this $V \geq V^0$.

Thus, all in all, I have derived the following (by combining the previous lemmas and propositions):

Theorem 1 *Supposing that $\alpha_{JH} > J_d^{SS}$, and condition (34) is satisfied, the optimal contract consists of two steps, one to p_l (while firing) at \tilde{t}_1 , and another one to p_h at \tilde{t}_2 (without dismissal), where $\tilde{t}_2 > \tilde{t}_1$.*

The conditions basically require that competition of other firms is not so strong that a worker who is unsure whether he will be fired will opt for an outside offer within no time. In this case, the firm will have to tell him right away by a wage of at least p_l . On the other hand, it also cannot be the case that the worker is almost completely sure that he is a high type, in which case he will almost completely disregard to dismissal risk. In this case, the firm might find extreme backloading the optimal policy. And again, the third, perhaps simplest condition is that it must be in the ex ante joint interest of the match to fire the low productivity worker. If the cost of dismissal is too high, for example when the productivities differ not by much, then it is best to never jump above p_l , and keep the lower productivity worker around.

Note that proposition 3 and 4 gives us an algorithm to solve for the optimal contract. We need to go backwards, from the time of the last jump up. Proposition 3 tells us at which productivity this end point is. Proposition 4 tells us candidate switching times. Each time the $\dot{\Pi}/\dot{V}$ ratio are equal at a (Π, V) , it could be optimal to switch.

It is straightforward to generalize proposition 4 to the case with more than two productivities. All that needs to be done is to generalize the notation, and check the switching conditions with all possible steps, at each (Π, V) . With abuse of notation, let Π_j be the current value of profit, and V_j the current value for the worker, at the switching time, before the dismissal decision. After the wage has gone up from p_{j-1} to p_{j+k-1} , these values equal

$$\Pi_{j+k} = \frac{1 - H(p_j)}{1 - H(p_{j+k})} \Pi_j, \text{ and } V_{j+k} = \frac{1 - H(p_j)}{1 - H(p_{j+k})} V_j - \frac{H(p_{j+k}) - H(p_j)}{1 - H(p_{j+k})} V^u.$$

Proposition 4 (b) *At an optimal contract with n productivities with switching times $\{t_1, \dots, t_n\}$, $0 \leq t_{i-1} \leq t_i \leq t_{i+1}$ (where $t_i = t_{i+1}$ denotes a step of two productivities at once), it necessarily needs to hold that, for a single-productivity step:*

$$\frac{\dot{\Pi}_j(\Pi_j, V_j)}{\dot{V}_j(V_j)} = \frac{\dot{\Pi}_{j+1}(\Pi_{j+1}, V_{j+1})}{\dot{V}_{j+1}(V_{j+1})} \quad (35)$$

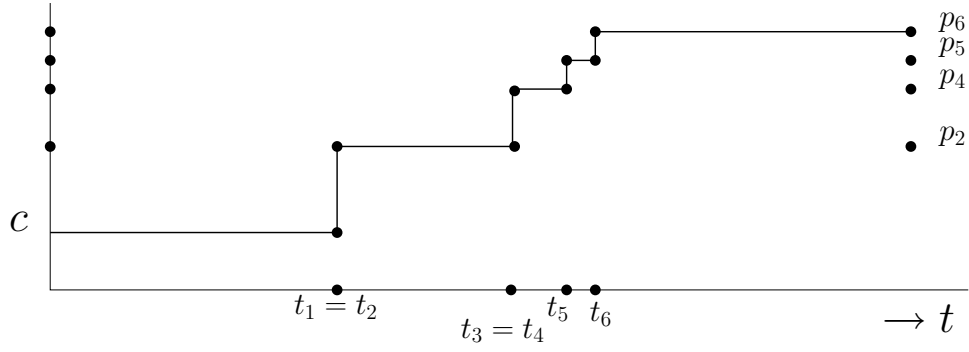


Figure 3: A multi-step contract

while for a multiple-productivity step, from j to $j+k$, $k > 1$, it holds that

$$\frac{\dot{\Pi}_j(\Pi_j, V_j)}{\dot{V}_j(V_j)} = \frac{\dot{\Pi}_{j+k}(\Pi_{j+k}, V_{j+k})}{\dot{V}_{j+k}(V_{j+k})}, \quad (36)$$

while for the intermediate productivities the following holds,

$$0 \geq \frac{\dot{\Pi}_{j+l}(\Pi_{j+l}, V_{j+l})}{\dot{V}_{j+l}(V_{j+l})} \geq \frac{\dot{\Pi}_{j+k}(\Pi_{j+k}, V_{j+k})}{\dot{V}_{j+k}(V_{j+k})}, 1 \leq l < k. \quad (37)$$

The proof for this more general case follows from the analysis in Appendix B.

5 Equilibrium in the Labor Market

Now suppose there is a continuum of firms of measure 1, and an identical measure of workers. Each firm will post a wage contract given the wage contracts of the other firms, and the steady state distribution of workers and their current values. The match quality can be either high or low. Firms that post wage contracts which are only profitable for workers in good quality matches are allowed to tell this to the worker, before he quits his old job (these firms would not have any incentive to lie about it). However, contracts are posted before the match-quality is revealed, and cannot be made contingent on the realized productivity.

After a worker meets a firm, accepts the offer, he will have a promised value that changes over time. Equilibrium in the labor market is defined as follows:

Definition 2 (Market Equilibrium) A Market Equilibrium is given by a list of functions and values, such that

1. Posted wage-tenure contracts $w(\cdot)$, continuation values V , Π , and the continuation decisions satisfy the conditions of a within-match perfect equilibrium (optimality and rational expectations), given that the initial value for the worker must be V_0 ; ex ante profit, $\Pi^{\text{ex ante}}(V_0)$, follows from the fundamentals and these decisions.
2. $F(V_0)$ is the distribution of posted (starting) values V_0
3. Optimal job search for the unemployed, implying V^u is given by

$$\delta V^u = b + \lambda \int_{V^u} (V_0 - V^u) dF(V_0)$$

4. $G(V)$ is the steady state distribution of current values of workers, consistent with the optimal turnover and dismissal decisions, and optimal wage-tenure contracts (and exogenous turnover).
5. for every initial value V_0 that is in the support of $F(V)$ it should hold that it is profit maximizing:

$$G(V_0)\Pi^{ex\ ante}(V_0) = \bar{P};$$

for any other V

$$G(V)\Pi^{ex\ ante}(V) \leq \bar{P}$$

Note that now each firm will have to choose a starting V , given the distribution of steady state values of workers, thus trading off acceptance with profitability. This distribution of steady state values should result from the posting of contracts by each firm, and the turnover decisions, given these contracts. Again, following the arguments in the previous section, we can concentrate on step contracts of the form studied before.

Optimal Wage-Tenure Schedules: Recap for general distributions *There always exists a step contract that is optimal. - For any contract that is optimal, there exists a (increasing) step contract that gives the same value to the worker.*

Let us look at those contracts that start of at wage p_l versus contracts that start at wage c :

Lemma 5 *Let v_{p_l} be an initial value contract posted that starts at p_l , let v_c be an initial value contract posted that starts at c . In equilibrium, $v_c \leq v_{p_l}$. Moreover, if a $v_c > V_L$ is offered in equilibrium, then it is profitable to offer $v_{p_l} > v_c > V_L$.*

Proof Suppose not. Let us first consider the case where the highest contract value offered by the p_l -firm, \bar{v}_{p_l} is lower than the highest value offered by the c -firm, \bar{v}_c . Then it must be that $P_h(\bar{v}_c) \leq P_h(\bar{v}_{p_l}) = P_l(\bar{v}_c)$. But, we can show that $P_h(\bar{v}_c) > P_l(\bar{v}_c)$: since it is the highest contract offered, the quit rate and exogenous match breakup rate together equal δ . Moreover, It needs to be the case that \bar{v}_c is feasible: i.e.

$$V^u + \alpha(V_H - V^u) > \bar{v}_c. \quad (38)$$

Because $\bar{v}_{p_l} \geq V_L$, we also must have

$$\bar{v}_c \geq V_L; \quad (39)$$

these two latter equations give us everything we need for the proof.

Let us first write the low contract guaranteeing \bar{v}_c :

$$\bar{v}_c - V^u = \frac{(c - \delta V^u)}{\delta}(1 - e^{-\delta T}) + e^{-\delta T} \alpha \frac{p_h - \delta V^u}{\delta}$$

Now the expected profit of the firm offering the high contract is given by (ignoring the identical acceptance probabilities):

$$P_h = \alpha \left[\frac{p_h - \delta V^u}{\delta} - \frac{(c - \delta V^u)}{\delta}(1 - e^{-\delta T}) - e^{-\delta T} \alpha \frac{p_h - \delta V^u}{\delta} \right];$$

rewriting this yields

$$P_h(\bar{v}_c) = \alpha \frac{p_h - c}{\delta}(1 - e^{-\delta T}) + \alpha(1 - \alpha)e^{-\delta T} \frac{p_h - \delta V^u}{\delta} \quad (40)$$

Now, given (39), we have

$$\begin{aligned} \bar{v}_c - V^u &= \frac{(c - \delta V^u)}{\delta} (1 - e^{-\delta T}) + e^{-\delta T} \alpha \frac{p_h - \delta V^u}{\delta} > \frac{p_l - \delta V^u}{\delta} \\ &\Leftrightarrow \\ e^{-\delta T} \alpha \frac{p_h - \delta V^u}{\delta} &> \frac{p_l - c}{\delta} (1 - e^{-\delta T}) + e^{-\delta T} \frac{p_l - \delta V^u}{\delta} \end{aligned} \quad (41)$$

Now, putting (42) and (41) together, we get

$$\begin{aligned} P_h(\bar{v}_c) &= \alpha \frac{p_h - c}{\delta} (1 - e^{-\delta T}) + \alpha(1 - \alpha) e^{-\delta T} \frac{p_h - \delta V^u}{\delta} \\ &> \alpha \frac{p_h - c}{\delta} (1 - e^{-\delta T}) + (1 - \alpha) \frac{p_l - c}{\delta} (1 - e^{-\delta T}) + e^{-\delta T} \frac{p_l - \delta V^u}{\delta} \\ &> \frac{(1 - \alpha)p_l + \alpha p_h}{\delta} (1 - e^{-\delta T}) = P_l(\bar{v}_c) \end{aligned} \quad (42)$$

Thus, we have established that the highest value, in an equilibrium with both c, p_l contracts, is offered by a p_l -firm. Now, suppose that there exists a $v_{p_l} < v_c < \bar{v}_{p_l}$, where

$$v_c = \sup\{v | v \text{ posted by a } c\text{-firm}\}.$$

Then $P_c(v_c) \geq P_{p_l}(v_c)$, or since acceptance decisions are the same: $\pi_c(v_c) \geq \pi_{p_l}(v_c)$, while at the same time $P_{p_l}(v_{p_l}) \geq P_c(v_{p_l})$. Now, look at $\frac{\dot{\Pi}}{\dot{V}}$:

$$\left(\frac{\dot{\Pi}}{\dot{V}}\right)_c(v_c, \pi_c(v_c)) = \frac{[\delta + \lambda(1 - F(v_{p_l}))]\pi_c(v_c) - \alpha p_h - (1 - \alpha)p_l + c}{[\delta + \lambda(1 - F(v_{p_l}))]v_c - c - \lambda \int \hat{V} dF(\hat{V})}$$

If we take $(v_c, \pi_{p_l}(v_c))$ as a starting point to go backwards, then

$$\left(\frac{\dot{\Pi}}{\dot{V}}\right)_c(v_c, \pi_{p_l}(v_c)) < \left(\frac{\dot{\Pi}}{\dot{V}}\right)_{p_l}((v_c), \pi_{p_l}(v_c)) < 0,$$

and also for every (v, π) that are reached by taking this system of differential equations backwards. Then it must be that at v_{p_l} , $\pi_c > \pi_{p_l}$. Now start the c -firm from $\pi_c(v_c) \geq \pi_{p_l}(v_c)$, then $\pi_c(v_{p_l}) > \pi_{p_l}(v_{p_l})$, contradicting the optimality of the p_l and c policies (the c -firm can increase its profit by offering a value v_{p_l}).

Therefore, it must be the case that in any labor market equilibrium, $v_c \leq v_{p_l}$. ■

This lemma ensures that the values offered in the market by firms are segmented into two sets, without overlap. Lower values starting with wage c , and higher values with wage p_l .

Let us now look at the steady state distribution of current values for workers. In steady state, the inflow of workers into a certain value equals the outflow. In the setting of this model the derivation is complicated by the fact that -in principle- it could be that firms offer wildly different wage schedules, e.g. jumps from one wage level to the other could occur at different values v . Thus, we have to start from the position that there are no baseline contracts (by the latter, we mean that every value offered corresponds to a different position on a wage schedule where jumps always occur at the same value). Call $\delta(v_c)$ the flow into unemployment from workers which previously had a current value v_c and higher. Call $\gamma(v_c)$ the outflow of employed workers to a part of their wage schedule with p_l -wages or p_h -wages (with a dismissal of workers in bad matches

associated with the raise), from a value of v_c or less. For these workers, wages were raised, they turned out to be in a productive match, and hence their current value has jumped. Let us define, also, $F_c(v_c)$ as the *mass (not distribution)* of firms offering a c -contract and posting a value v_c or less; $F_{p_l}(v_{p_l})$ the *mass* of firms offering a p_l -contract with a value v_{p_l} or less, and $[1 - F(v)] = F_c(\bar{v}_c) - F_c(v) + (F_{p_l}(\bar{v}_{p_l}) - F_{p_l}(v))$ by lemma 5. Call $F_c(\bar{v}_c) = F_1$, $F_{p_l}(\bar{v}_{p_l}) = F_2$. Moreover, with abuse of notation, let $(1 - \tilde{F}) = F_c(\bar{v}_c) - F_c(v) + \alpha(F_{p_l}(\bar{v}_{p_l}) - F_{p_l}(v))$. Likewise, let us define by $G_c(v)$, the *mass* of workers that is currently unemployed or receiving a c or p_l wage and do not know what the quality of their match is; and, $G_{p_l}(v)$ the mass of workers who is currently receiving a p_l or p_h , knowing that they are in a good match. $\tilde{G}(v)$ is the mass of workers with current value of v or less.

Then, we can derive the following about the steady state distribution

Lemma 6 *The distribution of workers over current values is given by:*

$$U = \frac{\delta + \delta(V^u)}{\delta + \lambda(F_1 + \alpha F_2)}.$$

For those workers that do not know what the quality of their match is or that are unemployed, the following relation holds (as a function of their current value v_c):

$$G'_c(v_c)\dot{V}_c(v_c) = \delta + [\delta(v_c) - \gamma(v_c)] - (\delta + \lambda(1 - \tilde{F}(v_c)))G_c(v_c) \quad (43)$$

For those workers that do know the quality of their match and, the steady state values of the mass is given by the following relation:

$$\begin{aligned} G'_{p_l}(v_{p_l})\dot{V}_{p_l}(v_{p_l}) &= \alpha\lambda \int_{\underline{v}_{p_l}}^{v_{p_l}} G_c(v)dF(v) - [\delta + \lambda(F_{p_l}(\bar{v}_{p_l}) - F_{p_l}(v_{p_l}))]G_{p_l}(v_{p_l}) \\ &\quad + \gamma((1-\alpha)V^u + \alpha v_{p_l}) \end{aligned} \quad (44)$$

where $\bar{\gamma}$ is the total flow of workers from any position in the wage schedule where they do not know their quality, to a position in the wage schedule where they do know that they are in a good quality match. (Note: $G_c(v_{p_l}) = G_c(\bar{v}_c)$, $v_{p_l} > \bar{v}_c$.)

Proof To see this, consider the flows during a small interval of time Δt ; During this interval of time, the value for the worker would change by $\Delta V(\Delta t)$. A worker would optimally accept all offers that offer a higher value

$$\begin{aligned} (1-G(V - \Delta V))\Delta t\delta + \delta(V - \Delta V)\Delta t \\ - (1 - \delta\Delta t)(\lambda\Delta t[1 - \tilde{F}(V - \Delta V)])G_c(V - \Delta V) \\ - \gamma(V - \Delta V)\Delta t - (1 - \delta\Delta t)(G_c(V) - G(V - \Delta V)). \end{aligned} \quad (45)$$

Dividing by Δt , taking $\Delta t \rightarrow 0$, and using the fact that the distribution is in steady state, gives us

$$G'_c(v_c)\dot{V}_c(v_c) = \delta + [\delta(v_c) - \gamma(v_c)] - (\delta + \lambda(1 - \tilde{F}(v_c)))G_c(v_c)$$

An argument along very similar lines establishes the same equation for G_{p_l} . ■

The next lemma establishes that there is always only one value (and one profit conditional on acceptance) associated with c -wage schedules, and p_l -wage schedules, for the case that there is no intermediate step in the wage contract.

Lemma 7 All firms that offer a starting wage of c , offer the same value in equilibrium; likewise for those offering p_l if there are no jumps to values $V_L < v < V_H$. Call F_1 the mass of firms posting a c -wage schedule, F_2 the mass of firms offering a p_l -wage schedule.

(1) For firms offering starting wage c , profits are given by $\dot{\Pi}_1^-(0) = 0$, or

$$\pi_c = \frac{(1 - \alpha)p_l + \alpha p_h - c}{\delta + \lambda(F_1 + \alpha F_2)} \quad (46)$$

(2) If there is no inflow from wage schedules starting at c into values $V_L < v$, there is a unique profit for p_l -firms.

Profit (conditional on acceptance) is given by:

$$\pi_{p_l} = \frac{p_h - p_l}{\delta + \lambda \alpha F_2} \quad (47)$$

(3) In case there is a jump to value V_H , we know that

$$G(v_{p_l})\pi_{p_l} \geq G(V_L) \frac{p_h - p_l}{\delta + \lambda \alpha F_2}$$

Moreover, here also it cannot be the case that an continuum of values is offered, or that there are masspoints at two different values, for p_l firms.

(4) If there is an inflow at (a set of) values V , such that $V_L < V < V_H$, it cannot be the case that the inflow is only at \underline{v}_{p_l} (the lowest value offered for p_l -wage schedules). Neither can it be that a continuous distribution of values of p_l -schedules is offered. Therefore, for the lowest value it still should hold that

$$\pi_{p_l} = \frac{p_h - p_l}{\delta + \lambda \alpha F_2}$$

Proof Consider the c -wage schedules first. For a v to give the maximum expected ex ante profit, it must be that case that

$$\frac{d^- P(v)}{dv} \geq 0, \frac{d^+ P(v)}{dv} \leq 0,$$

if ex ante profit is differentiable at v , it must be the case that

$$P'(v) = 0$$

Then,

$$P'(v) = G'_c(v)\pi(v) + G(v) \frac{\dot{\Pi}_c(\pi, v)}{\dot{V}_c(v)} = 0$$

substituting in from (43), we have

$$P'(v) = \frac{(\delta + [\delta(v_c) - \gamma(v_c)])\pi(v) - ((1 - \alpha)p_l + \alpha p_h - c)G_c(v)}{\dot{V}_c(v)}, \quad (48)$$

and for the second derivative

$$\text{sign } P''(v) = \text{sign} \left\{ (\delta'(v) - \gamma'(v))\pi(v) + (\delta + [\delta(v_c) - \gamma(v_c)]) \frac{\dot{\Pi}}{\dot{V}} - ((1 - \alpha)p_l + \alpha p_h - c)G'(v) \right\}. \quad (49)$$

There are three cases possible: (i) all firms with a starting wage of c offer the same value; (ii) there is a continuum of values offered by these firms; and (iii) firms offer two or more values, both values have a positive mass of these firms associated with them. We will rule out case (ii) and (iii) for (i). Consider case (ii): it must be that $P'(v) = 0$ along the continuum. This means that $\frac{\dot{\Pi}}{\dot{V}} < 0$. Now plug this into (49), and find that $P''(v) < 0$. But this contradicts that $P'(v) = 0$ on this interval (which would imply $P''(v) = 0$ as well). Likewise, consider the case of the two mass points, $v_1 < v_2$. If both give equal profit, it must be that $\pi(v_1) > \pi(v_2)$. We can show that $\frac{\dot{\Pi}}{\dot{V}}$ is monotonic. Then either $\frac{\dot{\Pi}}{\dot{V}} < 0$ for every $v \in (v_1, v_2)$, or for every $v \in (\bar{v}, v_2)$. In the first case, see that $P''(v) < 0$ contradicting equal profits. In the second case, there is a strictly profitable deviation to \bar{v} , thus contradicting optimality. Hence, all c -schedules have the same value. Then the only people that are hired are the unemployed. Going back to (48), $P'(v_c) = 0$ implies that

$$\pi_c = \frac{(1 - \alpha)p_l + \alpha p_h - c}{\delta + \lambda(1 - \tilde{F}(V^u))}$$

Note a couple of important things: profits are determined by instantaneous profits, and the extent of competition alone, not by the *values* which are chosen by firms who are posting other values than v_c . Competition is in a sense local, in that it does depend on the number of firms posting at v_c . Moreover, even though there is a mass of firms posting at v_c , one can check that the left-derivative of the ex ante profit function $P(v_c)$ equals the right-derivative. Note also that profit is given by the following equality:

$$\dot{\Pi}(\pi, v' < v_c) = [\delta + \lambda(1 - \tilde{F}(V^u))]\pi - (1 - \alpha)p_l - \alpha p_h + c = 0;$$

this is intuitive, since acceptance are the same for values $v < v_c$, it should be the case that the firm wins no less than it loses. Case (2) follows case (i) completely¹⁸.

Let us focus on case (3). Again, there is no jump up for any value before V_H . We have to show that it cannot be that there are mass points at two or more values for p_l -firms, or a continuum of values. Suppose, first, that there is a continuum of values offered. Let us look what this means for $P'(v)$:

$$P'(v) = \frac{\alpha\lambda \int G_c(v)dF(v)\pi(v) - (p_h - p_l)G_2}{\dot{V}_{p_l}} + \frac{\delta\pi(v) - (\delta + \alpha\lambda F_2)G_1\pi(v)}{\dot{V}_c} + \frac{(\delta + \alpha\lambda(1 - F(v)))\pi(v)G_1 - (p_h - p_l)G_1}{\dot{V}_{p_l}} \quad (50)$$

Now, knowing from case (i) that $\dot{\Pi}_{p_l}(\pi, v) < 0$ and $\dot{V}_{p_l} < \dot{V}_c$, we can establish from the equation above that for the first value offered in the continuum, $P'(v) < 0$, as the two terms on the RHS of the first line drop out. Now consider the case where there are two mass points: optimality of the first value v_1 dictates that $\dot{\Pi}_{p_l}^-(v) < 0$ (otherwise, it would be profitable to deviate to a value before v). Call the mass of the first mass point F_{21} and the second mass point F_{22} . Now, the first order condition reduces to

$$P'(v) = \frac{\dot{\Pi}_{p_l}^-(v)}{\dot{V}_{p_l}}G_c - \frac{(p_h - p_l)G_{p_l}}{\dot{V}_{p_l}} + G'_c(v)\pi_{p_l}(v)$$

¹⁸Note that in this case $\delta(v), \gamma(v) = 0$.

The sign of the second derivative is negative: each term has a negative derivative; in the last term rewrite $G'_c(v)$ as $\frac{\delta - (\delta + \alpha\lambda F_{22})}{\dot{V}_c} \pi$; see that the derivative is unambiguously negative. For the first two terms factor out the \dot{V}_{p_l} -term. At the optimum the remaining term $(\dot{\Pi}_{p_l}^-(v)G_c - (p_h - p_l)G_{p_l})$ must be negative. Since the derivative of the remaining term is also negative, the overall term $P'(v)$ can be shown to be negative. This means that a firm posting at the second mass point will make a positive profit from deviating to the first mass point.

The condition for a mass point is that no firm will want to deviate to another p_l contract. Taking the the left and right-derivative of $P'(v)$ at a mass point, we have:

$$P^{+'}(v) = \frac{[\delta - \delta G_c(v)]\pi(v)}{\dot{V}_c} + \frac{\left((\delta + \alpha\lambda F_2)\pi(v) - (p_h - p_l)\right)G_c(v)}{\dot{V}_{p_l}}; \quad (51)$$

and

$$P^{-'}(v) = \frac{[\delta - (\delta + \alpha\lambda F_1)G_c(v)]\pi(v)}{\dot{V}_c} + \frac{\left((\delta + \alpha\lambda F_2)\pi(v) - (p_h - p_l)\right)G_c(v)}{\dot{V}_{p_l}}. \quad (52)$$

Note that the last equation is unambiguously negative, for any $v > V_L$. From this it follows that in the equilibrium either ex ante profit is equal to

$$G_c(V_L) \frac{p_h - p_l}{\delta + \alpha\lambda F_2}.$$

Hence, it must be that ex ante profit is given by those (π, v) that satisfy both equations in the sense that there is no incentive to deviate to a v' before the v chosen (i.e. the profit at which (52) equals zero gives a weakly lower profit, while (51) at v is also not larger than zero. Note however that (51) is negative, even at V_L :

$$\begin{aligned} P^{+'}(v) &= \frac{[\delta - \delta G_c(v)]\pi(v)}{\dot{V}_c} + \frac{\left((\delta + \alpha\lambda F_2)\pi(v) - (p_h - p_l)\right)G_c(v)}{\dot{V}_{p_l}} \\ &< \frac{[\delta - \delta G_c(v)]\pi(v)}{\dot{V}_{p_l}} + \frac{\left((\delta + \alpha\lambda F_2)\pi(v) - (p_h - p_l)\right)G_c(v)}{\dot{V}_{p_l}} \\ &\leq \frac{(\delta\pi + \alpha\lambda F_2 G_1 \pi) - (p_h - p_l)G_1}{\dot{V}_{p_l}}, \end{aligned} \quad (53)$$

where the numerator is negative, because of the decreasing profit for the c -schedules. Hence, all p_l firms cluster at V_L , and their profit is given by the above equation¹⁹.

For the last case (4): it cannot be that in equilibrium, c -wage schedules jump up to v_{p_l} : this would be equivalent in profit to having a mass point of a c -policy at $(1 - \alpha)V^u + \alpha v_{p_l}$. Hence, the c -policies have to jump up later. A firm has no advantage of running up the value higher than V_L before jumping, and if more than a zero mass of firms would do this, there would be an incentive for other firms to deviate to p_l policies. This means that competition between p_l -policies is local. It also means that $\left(\frac{\dot{\Pi}}{\dot{V}}\right)_{p_l} = \left(\frac{\dot{\Pi}}{\dot{V}}\right)_c$ for $v \geq v_{p_l}$ (if unequal, the jump would either occur immediately, contradicting the previous statement, or never). ■

¹⁹For this lemma to cover all cases relevant, we still have to consider the intermediate case that only some firms give a wage schedule higher than V_L , whereas other firms stick to a one-step c -schedule.

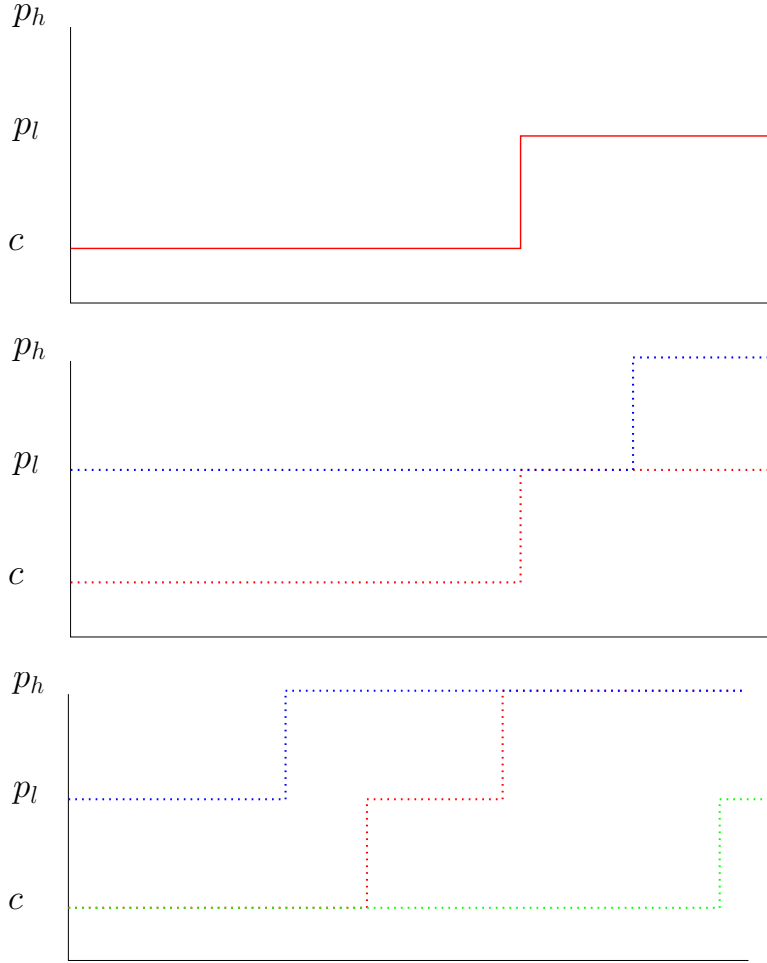


Figure 4: Wage Schedules (with increasing competition, from top to bottom)

Corollary 1 *There is never a jump up to p_h without p_l -firms; likewise, there are never p_l -firms without c -firms.*

Proposition 5 *Depending on the degree of competition, there are multiple forms that an equilibrium can take:*

1. *low degree of firm competition: if $\left(\frac{\delta}{\delta+\lambda}\right)^2 > \frac{\alpha(p_h-p_l)}{\alpha p_h+(1-\alpha)p_l-c}$, then there exists a wage schedule starting at c , given value v_c , and no wage schedule starting at p_l .*
2. *If $\left(\frac{\delta}{\delta+\lambda}\right)^2 < \frac{\alpha(p_h-p_l)}{\alpha p_h+(1-\alpha)p_l-c}$, but the ex ante expected surplus of a constant p_h and a p_l policy are not too close.*
3. *If competition is high enough, λ s.t. $v_h > (1-\alpha)V_H + \alpha V^u$, then there is an unique equilibrium (aggregate) outcome in the labor market, with a multi-step contract.*

The proof is in the appendix.

6 Discussion

6.1 Discussion of the Optimal Wage Schedule

In the section before the previous section, I have derived the optimal wage schedule. I have shown that if the right conditions are satisfied, the optimal contract comes in two steps, with the first step below the high productivity level, and the low-productivity worker would be fired at the *first* step.

In this case, the firm chooses not to reveal information right away. It prefers to keep the worker in the low-productivity match around for some time. Because the worker in the low match does not know what type he is, he is not leaving faster than a worker in a high-productivity match. Thus, initially, the firm prefers to pay the lowest wages possible, exploiting both the good and bad productivity employees (but, in a sense, the bad worker more, because the firm will never pay the worker in a low-productivity match more than c).

However, if the firm would engage in extreme backloading (where the wage would jump directly from c to p_h), the worker would discount the promised wage increase significantly more than in the gradual case, because of the uncertainty about the worker's retention prospect. Even more so, it would, up until the moment of the jump, also lead good workers to accept outside offers which they would have been better off rejecting, and the low-productivity workers would reject offers they would have been better off taking. This is costly for the worker, it decreases the value of being in the relationship and, because this leads to decreased participation of the worker, it is costly for the firm too.

Therefore, it is optimal for the firm to commit to an early wage jump to a level smaller than p_h . If the worker is still there after the jump, he can infer that he must be in a good match. In the two-productivity case, this means that he is certain that the firm will keep him, and eventually will pay him p_h . As a result, the worker in a good match is a lot less willing to leave the firm following the jump, while the firm is still able to make a profit from his service over some time interval.

The model captures the spirit of up-or-out contracts, with p_h and p_l interpreted as *cut-off* productivities. The firm commits in advance to a wage schedule, and a 'tenure' time with a wage increase. Up until that time, the firm keeps employing the low and the high-productivity worker, as the low-productivity worker is still useful at the given wage. (Think e.g. of law firms or accounting firms, where lawyers or accountants who are not that good are still productive, but just not good enough to be made partner, or of assistant professors prior to the tenure decision.) At the time of the 'tenure decision', the match quality is revealed (through the pay raise), even though the wage is not hiked up to maximum possible level. A good worker will now be tied more to the firm; he will no longer be easily tempted to leave for other firms.

As noted before, the contract is set in a second-best world, as I assume that the worker cannot see what he is worth to the firm, and the firm cannot see what offers the worker is getting. I assume that the firm does not have a cost-effective mechanism at its disposal that can make the worker truthfully reveal his outside offer, so this non-observability has a bite. As a result, the wage contract tries to approach the first-best by rewarding duration, but in a way that also involves mitigating the adverse impact of the participation decisions of the firm. Thus, wages will grow at an early stage, and the participation risk of the firm, which is based on a persistent factor, is resolved then. This means that whenever the worker becomes more certain about the participation of the firm, the firm can engage in more cost-effective backloading.

All in all, there are a lot of strategic forces at work that shape the growth of wages over tenure.

If, on the other hand, everything were observable, one would be in a first best world and contracts would take a simple form: the firm could take all the surplus until the worker got an outside offer, and then simply match whatever the competitor wants to pay. This wage would be paid until a lower production shock might force the wage downward while preserving the match. In the case with productivity shocks i.i.d. across firms and time, matches only break up when a firm making the outside offer has a higher productivity than the firm the worker is currently in, or the productivity shock is so bad that a worker who would be paid that productivity level would prefer to be unemployed instead. This is the case that Postel-Vinay and Turon (2006) study. With firms that are subject to the same distribution of shocks, this leaves out not only the ex ante concern about participation, but also the interdependence of participation decisions. In my model, it is the interdependence of participation decisions that is a major cause and determinant of the wage growth pattern.

6.2 Future Research

An important assumption in the paper is that the firm commits to a ‘simple’ wage-tenure contract: it specifies a single wage for each period that the worker is employed at the firm. Many firms have standardized wage contracts, or a standardized tenure track, so this assumption is not without realism. Moreover, this assumption allows us to make the trade-off in terms of participation very clear-cut: a wage increase over p_l would result in the dismissal of the p_l worker. Not every contract has to be so stark. I could investigate alternative behavior during the match. Two issues are particularly interesting: what would the contract look like if the firm could keep the worker in the low productivity match, instead of firing him? And, at what time would the firm want to tell a worker he is good if they could renegotiate the contract during the relationship? The general framework I constructed above allows these questions to be addressed as well. In that sense, the full commitment contract is the first step in this agenda, with more to come.

Another interesting dimension that can be relaxed is specification of uncertainty on the firm’s side. In the model, the uncertainty resolves at the beginning of the match, the worker just doesn’t know about the outcome. Wages are informative because the shock is persistent, but nothing requires persistence to take a once-and-for-all form. I can generalize this setup to include uncertainty that realizes *during* the match. As long as there is some degree of persistence of the productivity shock, wages will be informative. An interesting application would be the response of wage schedules to the amount of uncertainty associated with working for a firm.²⁰

In its current version, the model speaks to *qualitative* aspects of the data. Needless to say, a less stylized version of the model could aim at quantitative analysis as well. The list of topics to be addressed, apart from those mentioned above, also include other observations. For example, workers with higher tenure that are dismissed often incur a larger wage loss. The model could also address the pattern of job separations over the job relationship, which is hump-shaped with a slowly declining tail (one that is declining too slow to be adequately explained by pure learning

²⁰An interesting observation in this respect is that life-cycle wage profiles have flattened over the last decades (Kambourov and Manovskii 2004,2005) (Marcotte (1998) has found that the return to tenure has decreased over time, while at the same time, volatility of firm performance and profit has gone up (Comin and Phillipon (2005)). A theory set in the framework of bilateral uncertainty could provide a link between wage profiles and firm volatility, and could have something to say about turnover patterns as well (e.g. decreased tenure on jobs, not only because of shocks but because of changes in the labor market equilibrium). A further interesting dimension is that the patterns of job insecurity have changed as well, with the risk of layoffs shifting towards more senior workers; the model presented here would allow an investigation of whether changes in wage profiles could be accounted for by these shifts.

alone (Van der Ende and Teulings 2002), and the evolution of layoff hazards over job relationships as well.

7 Conclusion

In the paper, I have proposed a theory of long-term contracting that takes as a crucial element of the relationship between firms and workers the participation uncertainty of either party. The optimal contract was characterized, which trades off instantaneous profit with the need to retain the firm's workers, but in particular the workers in high-productivity matches. The resulting wage-tenure profile exhibits *gradual* growth over the course of the employment relationship, driven by (and responsive to) the bilateral nature of uncertainty. A direct implication of this model is that the uncertainty a worker attaches to his employment affects the optimal wage profile that the employer wants to give him over the course of the relationship. The model can be readily extended to study this effect in a market equilibrium as well. It can thus speak to important features of the data, such as wage growth with tenure, the decline of turnover and dismissals over an employment relationship, and the link between uncertainty properties of the economy and labor market outcomes.

As a final note, it is interesting to see that this alternative theory about wage growth has very different implications for allocative efficiency, than for example, human capital. In this model firms try, by backloading wages, to monopolize the worker over time. However, how effectively they can do this depends on the certainty about continued employment at the firm. A dynamic, or perhaps 'turbulent' economy might therefore have an additional benefit: it could induce firms to flatten wage schedules, which would make workers more free to move from place to place.

Appendix

A Proof of Propositions

Derivation of Equation (12) One can rewrite

$$\delta V^u = b + \lambda \int_{V^u} (\hat{V} - V^u) dF(\hat{V})$$

as

$$\delta V^u = b - \lambda(1 - F(\tilde{V}))V^u + \lambda \int_{V^u}^{\tilde{V}} (\hat{V} - V^u) dF(\hat{V}) + \int_{\tilde{V}} \hat{V} dF(\hat{V}).$$

Then, it follows that

$$V^u = \int_0^\infty (e^{-\delta t} \psi(t)) \left(b + \lambda \int_{\tilde{V}(t)} \hat{V} dF(\hat{V}) + \int_{V^u}^{\tilde{V}(t)} (\hat{V} - V^u) dF(\hat{V}) \right) dt.$$

Now use this formulation of V^u in $\Pi + V - V^u$, to get equation (12).

Proposition 1 *An optimal contract will have wages that are weakly increasing over time; the optimal quit decision therefore occurs at $w = p$. (Or, if wages jump up, at the first time that $w \geq p$.)*

Proof of the case with a continuous $H(p)$ Let us first consider the case that $H(p)$ is continuous. Suppose that there is a decrease in wages in the optimal contract. I want to construct a non-decreasing wage schedule that is preferred by both parties, strict by at least one of them.

Concretely, by a decrease in wages I mean that I can find (t_1, t_2) with $t_1 < t_2$, such that there exists a interval of times between t_{01} and t_{02} , with $t_{01} < t_{02} < t_1$, such that $\forall t_0 \in (t_{01}, t_{02}), w(t_0) > w(t), \forall t \in ((t_1, t_2))$.

For this proof, it is important to understand when dismissal (i.e. break-ups induced by the firm) take place. Hence the following lemma

Lemma 8 *A worker is never dismissed during a time interval that where are decreasing. Moreover, if wages increase continuously, then a worker is fired only at $w = p$. (Or, if wages jump up, at a time \hat{t} , where $w(t) \geq p$, whereas there was an interval $T = [\hat{t}, \hat{t}]$, such that $w(t) \geq p \forall t \in T$)*

Proof Denoting (t_1, t_2) to be an interval in which wages fall, it must be that there exists an time-interval around and including (t_1, t_2) in which no dismissals occur. This follows from the simple fact that a firm who was not willing to fire workers at t_1 , will for sure receive an positive, or less negative instantaneous flow of profits over the (t_1, t_2) interval.

In particular, At t_1 , $\Pi(t_1, p) \geq 0$. If $w(t) > p$, then expected profit $\Pi(t, p)$ increases during the decrease in wages, by the fact that $\Pi(t_1, p) \geq 0$ given the optimal decision $t(p)$. A layoff time, $t(p)$, during the time-interval of the decreasing wages interval contradict the greater than zero profit at the beginning. In words, the firm is paying $w(t) > p$ as an investment, to get a larger profit later. Since it was optimal to start investing, it is optimal to continue. Therefore, the firm should hang on to the worker. If $p > w$, then the firm is making profit, and there can be no reason to fire the worker.

It is straightforward to see that dismissals of a match that drew p , occur only at $w(\bar{t}) = p$, or at $w(\bar{t}) > p$ only if $\lim_{t \uparrow \bar{t}} w(t) < p$.²¹ By the above reasoning, they cannot occur in a decreasing wage-interval. Thus, wages must be increasing (weakly) at the time of any dismissal. Then, if they would occur sooner, a profit could be made by postponing, if they occur later, a profit could be made by dismissing earlier. ■

There are four cases: there is an interval of decreasing wages before anyone is fired (case 1); secondly, there is a interval of decreasing wages between the dismissal of two types (case 2). And finally, workers are fired initially, and wages decrease subsequently without any dismissals (case 3); and finally, there is a decreasing wage interval but there no layoffs at any time.

I prove the proposition by constructing an alternative (weakly) increasing wage schedule that yields at least as much profit to the firm, and at least as much value to the firm, for these three cases. The basic idea is the following: if one redistributes the wages in such a way that a) one backload wages as much as possible, without b) violating the firm's break-up condition, while c) keeping the initial values constant for both worker and firm *constant, given unchanged break-up decisions of workers*, then it follows that when one allows workers to change their breakup decisions given the new wage schedule, both workers and firms are better off. Workers by revealed preference, and firms because workers' revealed preference is to actually stay longer with the match. Since each productivity was profitable, and the wages are never so high as to induce a new breakup, this is better for the firm too.

□ **Case 1** Wages decrease on some interval, before t_f , where the first guy is fired. Note first that it must be that $t_f > c$, where c is the minimum wage the firm can pay. Let $\bar{\Pi}(p, t_f)$ be defined as

$$\bar{\Pi}(p, t_f) \equiv \int_0^{t_f} \psi(t)(p - w(t))dt,$$

and \bar{V}_{t_f} as

$$\bar{V}_{t_f} = \int_0^{t_f} \psi(t) \left(w(t) - b - \lambda \int_{V^u}^{\bar{V}} V dF(V) \right) dt$$

Note that I have defined \bar{V} here as the value for the worker above unemployment.

²¹I focus on wage-tenure contracts that are a.e. continuous, and a.e. differentiable in this paper. I show that they are increasing and bounded, so this property is natural.

Now, given that I keep $\tilde{V}(t)$ constant (given that the break-up decisions are also held constant!), to constraints for the new, backloaded wage schedule $w^b(t)$ are

1. $c \leq w^b(t) \leq p \forall t, 0 \leq t \leq t_f$
2. $\int_0^{t_f} \psi(t)w(t)dt = \int_0^{t_f} \psi(t)w^b(t)dt$

Find $t^{bs} = \left\{ t \mid \int_0^{t_f} \psi(t)w(t)dt = \int_0^{t^{bs}} \psi(t)cdt + \int_{t^{bs}}^{t_f} \psi(t)pdtdt \right\}$. This t^{bs} exists, otherwise it would have been optimal to fire p at $t = 0$.

Let the worker now choose optimally, given the new wage schedule. He is always able to copy his old acceptance schedule of outside options, and thus always able to obtain his old value. Any new choice is therefore an improvement for him. For the firm, note that the new wage schedule implies that current value $\tilde{V}(t)$ for worker is higher than before, at any $t > 0$. This means that the new $\psi^{bs}(t)$, given the new optimal responses of workers is always higher than before, i.e.

$$\bar{\Pi}^{bs}(p, t_f) \equiv \int_0^{t_f} \psi^{bs}(t)(p - w(t))dt > \bar{\Pi}(p, t_f) \equiv \int_0^{t_f} \psi(t)(p - w(t))dt$$

This will in fact hold for any p , and therefore also for the ex ante expected profit.

Hence, I have constructed a wage schedule that (pareto) dominates the old schedule with the decrease, and is strictly better for the firm.

□ **Case 2** Take an interval between two dismissals, with an sub-interval of decreasing wages in between. I will show that a constant wage will do better. The firm is indifferent between firing a worker at the beginning and at the end of the $(t_1, t_2]$ or $[t_1, t_2)$ interval, when this worker has a productivity p , associated with either the maximum of productivities fired before t_1 (in case $(t_1, t_2]$), or the minimum of the productivities fired after the time interval (in case $[t_1, t_2)$). This means that

$$\bar{\Pi}(p) \equiv \int_{t_1}^{t_2} \psi(t)(p - w(t))dt = 0$$

Now, let's replace the (t_1, t_2) -interval of original wage-tenure contract, by a constant wage $w = p$. This will yield zero profit to the firm, if productivity is p . Along the same lines as above, I can show that the worker will do better by this contract, and decide in strictly more cases to stay with the firm. Moreover, any firm with $p' > p$ will do strictly(!) better.

□ **Case 3** There is a decreasing wage schedule, strictly decreasing over some time, and all dismissals occur at the beginning. Again, look at the supremum of the productivities fired at the beginning, p . At this productivity is the firm is indifferent. Consider the alternative of a constant wage, p . Again, going through the same reasoning as above, this will give strictly more profit to firms, and weakly more value to workers.

□ **Case 4** Follows case 1 close, but now over an unbounded time interval. Take p to be the lowest productivity at which workers are fired. Clearly, at no point did profit for this productivity become negative. This means that at no point did the firm promise an amount to the worker that it could not pay unless it would raise wages above p . Hence, the problem reduces to finding \tilde{t} before which $w(t) = c$, after which $w(t) = p$, while providing the same value (holding constant match-breakup decisions of workers). This \tilde{t} exists, and by revealed preferences workers will do better, and by strictly increased match survival firms will do better.

Putting all pieces together: there cannot optimally be a strictly decreasing part of a wage-tenure contract. Hence, the first time $w = p$, or w jumps over p , the firm will disband matches with productivity p . Note that the strictness was derived under the assumption of a continuous distribution of $F(V)$ with full support. The proof can straightforwardly be adapted for the case without full support (in which these preferences will

be weak), and the statement is that there exists an increasing contract that is not dominated by any other contract.

Note that the last case basically provides an alternative proof of the (weak) optimality of step-functions as the optimal wage-tenure relation in the standard case in Stevens (2004). The proof is perhaps a bit longer (but mainly since one has to worry about firm-induced dismissals here), but it has a very intuitive ring to it.

■

Proof of the case with a discrete r.v. $\sim H(p)$ The different cases basically follow the same line; hence I pick one case to illustrate the difference. The other cases follow along the same lines. Let us focus on case 2 because continuity has most bite there.

Take again the time interval between two dismissals (t_1, t_2) , with a decreasing wage during some sub-interval of time. Now, one does not have the indifference in profit. In fact, when p is fired, then productivity right above it, p' still makes positive profit. However, I can once again construct a step contract that gives equal profits, given constant break-up decisions, the firm and the worker.

Note that I have always constructed flat or increasing profiles, and therefore the previously optimal firing decisions stay optimal. If instead one would make very large steps (first down to c , then up to the firing wage, this would also be a dominating contract, given the firing decision constant. However, the firing decision would adjust, and the contract with the changed firing decision would once again be dominated by an increasing contract. ■

There is one cautionary note: one can construct non-generic examples where there is exact indifference between the surplus maximizing policy with and without dismissals. In this case there are nonmonotone contracts that are optimal too. However, there is always a monotone contract that achieves the values.

Proposition 2 *In equilibrium, the firm breaks up a match with productivity p_i as soon as the wage w in the contract jumps up to p_i . Also, it is never optimal to break up matches with a productivity p_i at two or more different times.*

Proof In principle there could be four cases of continuation values (Π^-, V^-, J^-) at time of dismissal (incorporating the expectation of being dismissed), in relation to the values a match can achieve in expectation, without dismissals (Π_L, V_L, J_L) . The four cases are: 1) $J^- > J_L, V^- > V_L$, 2) $J^- < J_L, V^- > V_L$, 3) $J^- > J_L, V^- < V_L$, and 4) $J^- < J_L, V^- < V_L$. I want to show that in none of these cases is a dismissal preceded by a period of p_l . I can restrict ourselves to contracts to offer either p_l or c .

□ **Case 1** These cases share that $V^- > V_L$. I will show that $w = c$ is the best policy. Call w_s, V_s, J_s the wages and values of this policy, index by g some other policy. Then

$$w_s(t) \leq w_g(t) \Rightarrow V_s(t) > V_g(t)$$

by the same reasoning as in lemma 1. Note that as result of $w_g > w_s$ during some interval, the dismissal time for each policy is different. But at the jump time \tilde{t}_s , it must be the case that $V_g(\tilde{t}_s) < V^-$, and $J_g(\tilde{t}_s) < J^-$. As a result, take \tilde{J}_s, \tilde{J}_g , and find that $J_s(0) > J_g(0)$.

□ **Case 2** Take the $w_s = c$ policy, then, suppose $V_0 = V_L$. Find the step time, and find $J_s(0)$. If $J_s(0) > J_L$, the optimal policy is to have a wage equal c till \tilde{t} , at which one has J^-, V^- . If $J_s(0) < J_L$ (at V_L), the optimal policy is *not* to jump up with dismissal, instead it is to jump up to p_l without dismissals. At $J_s(0) = J^-$, then it does not matter. In this case, the firm *could* offer a wage p_l before dropping back to c . This is a non-generic case.

For case 3 and 4, I invoke a little lemma,

Lemma 9 *If the values at dismissal are V^-, J^- , there always exists a policy that gives values $V^{-'} > V^-$, and still achieve a surplus $J^{-'} > J^-$.*

Proof Look at (27). Call $J_H(V)$ the surplus when the worker has a time-invariant value V . Observe that $J_H(V)$ is nondecreasing in V until $V = J_H$. ■

□ **Case 3** $J^- > J_L, V^- < V_L$ with the appropriate change in policy after dismissal, from lemma 1, I can disprove the following: suppose it were optimal to have $w = p_l$ for some time. Then, I would have V decreasing over some interval. Call \bar{V} the highest V gets. Since $J^- > J_L$, J is increasing over the interval that V is decreasing. Moreover, the rate of the increase \dot{J} is faster than \dot{V} ($\dot{\Pi} > 0$ as $\Pi^- > \Pi_L$, and $V < V_L$). By the above lemma, one could also increase value at dismissal V^- to \bar{V} , without changing J^- (even improve it possibly). Hence profit falls by less under the last alternative, as under the decreasing V policy. So no time of p_l wages is offered.

□ **Case 4** $J^- < J_L, V^- < V_L$. Along the same lines as case 3, I can show that one needs to be on the upward-sloping part of the V profile. Trace back the surplus from the J_L, V_L , with $w = c$ to $V_0 = V^-$. If $J(0) < J^-$, the optimal policy is c till $V = V^-$, then dismiss. If $J(0) > J^-$, then the optimal contract would be a non-dismissal contract that jumps up to p_l , or a contract with a different $J^{-'}, V^{-'}$, such that $J(0) < J^{-'}$. ■

Proposition 4 *The conditions for the steps in a two-step contract are given by the following: going backwards in time, there is a step down from p_l to c , at time $\tilde{t}_2 - \tilde{t}_1$ when*

$$\frac{\dot{\Pi}^{ecv}(\alpha \Pi_h, \alpha V_h + (1 - \alpha)V^u)}{\dot{V}^{ecv}(\alpha V_h + (1 - \alpha)V^u)} = \frac{\dot{\Pi}^{cv}(V_h, \Pi_h)}{V^{cv}(V_h)} \quad (54)$$

at some Π, V , taking \dot{J}, \dot{V} backwards from $V_H = J_H$.

Proof Let us concentrate on the worker's value V first. The problem should hold the initial V_0 constant. Split (32) into two parts:

$$V_0 = \int_0^{t_1} e^{-\delta t} \psi(t) (w + \int_{\tilde{V}(t)} \hat{V} dF \hat{V}) dt + e^{-\delta t_1} \psi(t_1) V_1^-,$$

where V_1^- is the expected continuation value in the second stage. Likewise,

$$V_1^- = \alpha \int_0^{t_2 - t_1} e^{-\delta(t - t_1)} \psi(t - t_1) (w + \int_{\tilde{V}(t)} \hat{V} dF \hat{V}) dt + \alpha e^{-\delta(t_1 - t_2)} \psi(t_1) V_1 + (1 - \alpha)V^u.$$

A change in t_1 causes an offsetting change in t_2 , to keep V_0 constant. I solve this in two steps. First, a change in t_1 leads to a change in V_1^- . Then I calculate the change in t_2 to compensate for the change in V_1^- .

$$\frac{dV_1^-}{dt_1} = (w + \lambda \int_{\tilde{V}(t_1)} \hat{V} dF(\hat{V})) - (\delta + \lambda(1 - F(\tilde{V}^-(t_1)))V_1^-.$$

Notice that, since we have kept V_0 constant, we also have not changed any current value $\tilde{V}(t)$, notice that it equals $\tilde{V}(t_1)$ (which is not that surprising). The same trick can be repeated, for the second part, but now we go back from V_2 ; in other words, we keep V_2 and the time that we get there constant. So again, $\tilde{V}(t)$ is unchanged. For that purpose, let time go backwards:

$$V_2 = e^{\delta t_2} e^{\int_0^{t_2} \lambda(1 - F(\tilde{V}(\zeta))) d\zeta} V_1 - \int_0^{t_2} e^{\delta t} e^{\int_0^t \lambda(1 - F(\tilde{V}(\zeta))) d\zeta} (p_l + \lambda \int_{\tilde{V}(t)} \hat{V} dF(\hat{V})) dt$$

Then, dV_1/dt_2 yields the following:

$$\frac{dV_1^+}{dt_2} = (\delta + \lambda(1 - F(V_1^+)))V_1^+ - p_l - \lambda \int_{V_1^+(t)} \hat{V} dF(\hat{V})$$

The t_2 here is the difference between t_1 and t_2 before (because we relabeled the time). Thus, we have derived the following:

$$\frac{dt_2}{dt_1} = - \frac{\dot{V}_l(\alpha V + (1-\alpha)V^u)}{\alpha \dot{V}_h(V)},$$

noting that

$$\frac{dV_1^-}{dV_1^+} = \alpha.$$

We can repeat the same exercise for Π_h, Π_l :

$$\frac{dt_2}{dt_1} = - \frac{\dot{\Pi}_l(\alpha \Pi, \alpha V + (1-\alpha)V^u)}{\alpha \dot{\Pi}_h(\Pi, V)}$$

Now, optimization would require that we keep increasing t_1 as long as $0 > \left. \frac{dt_2}{dt_1} \right|_{\Pi} > \left. \frac{dt_2}{dt_1} \right|_V$. As long as this is the case, the change that would keep profit equal makes the worker better off; and a change that makes the worker at least as well off makes the firm better off. So, it is good to increase t_1 . Likewise, we could rewrite the problem in terms of the distance between t_2 and t_1 . Going backwards from J_H, V_H (which is easier to do, to keep track of Π and V), we would keep increasing the distance between t_2 and t_1 as long as $\left. \frac{dt_2}{dt_1} \right|_{\Pi} < \left. \frac{dt_2}{dt_1} \right|_V$.

The switching point occurs when $\left. \frac{dt_2}{dt_1} \right|_{\Pi} = \left. \frac{dt_2}{dt_1} \right|_V$, or when

$$\frac{\dot{V}_l(\alpha V + (1-\alpha)V^u)}{\alpha \dot{V}_h(V)} = \frac{\dot{\Pi}_l(\alpha \Pi, \alpha V + (1-\alpha)V^u)}{\alpha \dot{\Pi}_h(\Pi, V)},$$

or

$$\frac{\alpha \dot{\Pi}_h(\Pi, V)}{\alpha \dot{V}_h(V)} = \frac{\dot{\Pi}_l(\alpha \Pi, \alpha V + (1-\alpha)V^u)}{\dot{V}_l(\alpha V + (1-\alpha)V^u)}$$

■

For a derivation of this result in an alternative way, see below, where we set up the optimal contract problem as a so-called hybrid optimal control problem.

Proposition 5 *Depending on the degree of competition, there are multiple forms that an equilibrium can take:*

1. *low degree of firm competition: if $\left(\frac{\delta}{\delta+\lambda}\right)^2 > \frac{\alpha(p_h-p_l)}{\alpha p_h+(1-\alpha)p_l-c}$, then there exists a wage schedule starting at c , given value v_c , and no wage schedule starting at p_l .*
2. *If $\left(\frac{\delta}{\delta+\lambda}\right)^2 < \frac{\alpha(p_h-p_l)}{\alpha p_h+(1-\alpha)p_l-c}$, but the ex ante expected surplus of a constant p_h and a p_l policy are not too close.*
3. *If competition is high enough, λ s.t. $v_h > (1-\alpha)V_H + \alpha V^u$, then there is an unique equilibrium (aggregate) outcome in the labor market, with a multi-step contract.*

Proof The first thing to realize is that lemma 7 restricts the possibilities of wage schedules that can be offered in equilibrium to a great extent. For one, it says that under the conditions stated, there can be only one value offered for wage schedules that start at a wage c , or alternatively, at a wage p_l . This means that we only have to see (a) whether firms can be ex ante indifferent between a c - and a p_l -contract. Local competition determines the profit conditions, as once again can be seen from lemma 7 (i.e. profit depends on the parameters, and the mass of firms posting at a p_l or c wage), however, (b), the evolution of the wage schedule is in turn affected by the relative benefits of jumping up or not. Hence, we can derive three different cases: 1) the case where it is not profitable to offer p_l -wages. In this case it is not profitable to jump up. 2) the case where it is profitable to offer p_l -starting wages, but it is not profitable to jump up. Finally, 3), the case that it is profitable to offer p_l wages and it is profitable to jump up.

□ **Case 1:** *The case of a low degree of firm competition: if*

$$\left(\frac{\delta}{\delta + \lambda}\right)^2 > \frac{\alpha(p_h - p_l)}{\alpha p_h + (1 - \alpha)p_l - c},$$

then there exists a wage schedule starting at c , given value v_c , and no wage schedule starting at p_l .

If there is no other firm offering a starting wage at p_l level, ex ante profit for this policy is given by

$$P_{p_l} \equiv 1 \cdot \alpha \cdot \frac{p_h - p_l}{\delta}, \quad (55)$$

note that everybody accepts a wage offer (we can show that if there is no positive mass of firms offering p_l -starting wages, it is not optimal to jump up); hence every worker has a value $v \leq V_L$. And a wage $p_l(+\epsilon)$ yields a value $v \geq V_L$. The firm will accept the worker only in α of the cases, and the profit is will make is, conditional on having a worker of high output is the last term, on the very right of the above equation.) Given that all other competitors are at a c -wage, ex ante profit for those firms is given by

$$P_c \equiv \frac{\delta}{\delta + \lambda} \frac{(1 - \alpha)p_l + \alpha p_h - c}{\delta + \lambda} \quad (56)$$

Comparing the two ex ante profits, we must have that the former is lower than the latter to prevent any firm from offering p_l starting wages. Moreover, we can see that the surplus of a c -policy is lower than the surplus of a p_l policy: $\alpha(\Pi_{p_l} + V_L - V_u) < \Pi_c + V - V^u$, as long as $p_l/\delta > V^u$, because then

$$\alpha\left(\frac{p_h}{\delta} - V^u\right) < \frac{\alpha p_h + (1 - \alpha)p_l}{\delta} - V^u.$$

Note that the LHS is decreasing in λ , so as the competition parameter increases, the more profitable a p_l starting wage-policy becomes. See also that this inequality holds strictly, even if $P_{p_l} = P_c$. Hence, there is a region as in case 2, below.

□ **Case 2:** *If*

$$\left(\frac{\delta}{\delta + \lambda}\right)^2 < \frac{\alpha(p_h - p_l)}{\alpha p_h + (1 - \alpha)p_l - c},$$

but the ex ante expected surplus of a constant p_h and a p_l policy are not too close yet, the equilibrium will have a mass of firms posting c starting wages, ending at p_l . There will also be a mass of firms starting at p_l , with jumps to p_h after a certain time \bar{t}_2 .

Now, it is profitable for some firms to post p_l starting wages. Given that the expected surplus of jumping up is not high enough, by construction, all firms which offer c starting wages will never jump to p_h , and never dismiss workers. This means that, once again, the firms with p_l starting wages will get any unemployed that they accept, and also any worker for a c -firm that they accept. Equilibrium is found such that $G(\cdot)$ is in steady state, and equal expected profits are the case for p_l wage schedule, and the c schedule. Note that of the steady state $G(\cdot)$, two points are important: the mass of unemployed, and the mass of workers in a c -schedule,

$$U = \frac{\delta}{\delta + \lambda(F_1 + \alpha(1 - F_1))}, G_1 = [1 - G_2] = \frac{\delta}{\delta + \lambda\alpha(1 - F_1)}$$

The profits are given by

$$P_c = \frac{\delta}{\delta + \lambda(F_1 + \alpha(1 - F_1))} \frac{(1 - \alpha)p_l + \alpha p_h - c}{\delta + \lambda(F_1 + \alpha(1 - F_1))},$$

$$P_{p_l} = \frac{\delta}{\delta + \lambda\alpha(1 - F_1)} \frac{\alpha(p_h - p_l)}{\delta + \lambda\alpha(1 - F_1)}$$

Note that this implies that as λ increases, F_1 decreases: i.e. the competition among p_l wage schedules increases. Moreover, this implies that the value offered to workers with p_l wage schedules increases. Also, as

F_2 increases, the surplus of c -policies decreases, while the surplus of p_l policies stays constant. If λ is high enough, and $v_h > (1 - \alpha)V^u + \alpha V_H$ in equilibrium, *and the surplus of the p_l policy is equal to the surplus of the c -policy, then we are in case 3.*

□ **Case 3:** *If competition is high enough, λ s.t. $v_h > (1 - \alpha)V_H + \alpha V^u \equiv \bar{v}$, then there is an unique equilibrium (aggregate) outcome in the labor market, with a multi-step contract.*

Note first that it is not possible to have an equilibrium with surplus of p_l policies strictly higher than c -policies, and $v_h > \bar{v}$. This would mean that every c -schedule would jump up v_h . But then firms offering v_h make as much profit as offering a c -contract at $v = \alpha v_h + (1 - \alpha)V^u$ (there is no increased acceptance from offering a v_h contract, as all wage schedules jump from $v = \alpha v_h + (1 - \alpha)V^u$ to v_h). But then, according to lemma 7, this cannot be profitable. Hence, we cannot have this strict dominance of surplus. (A heuristical outline of the proof follows.) As λ has increased to the level that $J_{p_l} = J_c$, firms are indifferent between jumping up to p_l and p_h , with dismissal, and jumping up to p_l later without dismissal. As λ increases even further, we can prevent an increase (or even cause a decrease) in F_2 by having more and more firms with c -schedules jump up, to keep the equality between J_{p_l}, J_c . Note that, in this case we know that can we can go from too much F_2 ($F_2 = 1$), in case nobody jumps up, to $F_2 = 0$, by having every schedule jump up at v_h . Hence, since we can affectively manage G_1 continuously between strictly profitable, and strictly unprofitable, with intermediate jumps, we can construct G_1 that creates the indifference. Note that at any λ s.t. $J_{p_l} < J_c$, we do not have this freedom to vary the jump times continuously. ■

B The Optimal Contract as a Hybrid Optimal Control Problem

In this section I derive the conditions in proposition 4 as the solution of a hybrid optimal control problem, i.e. an optimal control problem that involves both discrete and continuous elements (see e.g. Branicky, Borkar and Mitter (1998), Sussmann 1999). In this derivation, for brevity, I will use the results of propositions 1-3, but these can be derived within the context of this optimal control problem as well (although the intuition behind the results is easier to see in the setting of the main text).

Following Doole (2005) closely, I will set up a multiple-phase optimal control problem, in which the switching times are also choice variables. Subsequently I will state the relevant theorem about the necessary conditions for optimality of this problem.²² Subsequently, I will phrase the problem in such a way that the theorem applies, and derive the optimality conditions. I will then show that the necessary conditions reduce to the conditions of proposition 4.

Define a multiple-phase switching system as $\Theta = \{T, X, \{f_i\}, U, \{F_j\}\}$, where the elements are given below.

- Let $T = \{t_1, \dots, t_n\}$ be a set of discrete controls that dictate the end of one phase and the beginning of the next. These we call the switching times. We restrict the order of the switches: $t_{i-1} \leq t_i \leq t_{i+1}$. The end-time t_n is freely determined.
- Let x an m -dimensional state vector in a continuous state where $X \in \mathbb{R}^m$,
- U is an open set of admissible controls. We restrict u to be continuous function within each phase. (From proposition 1-3, we know that the optimal function indeed must be continuous within each phase.)
- During the time interval between t_{i-1}, t_i , the vector field f_i is active, defining the law of motion for the state variables. We assume $f_i(x, u)$ is continuous in x and u ;
- F_j is the instantaneous payoff function to the firm, given states $x(t)$ and continuous control $u_j(t)$, when the firm is in phase j .

²²For related results in settings that are similar, but not identical, see Tomiyama and Rossana (1989) and Makris (2001). The setup of the problem follows Doole directly (section II. in his paper) and the theorem is proved in the appendix of his paper.

Thus, feasible controls for this problem are $\{T, \{u_i(t)\}\}$, satisfying the conditions above. An trajectory $\{T, x(t), \{u_j(t)\}\}$ is admissible if there exist feasible controls that yield the evolution of the state variable, given the laws of motion $\{f_i(x(t), u(t))\}$.

Problem B.1 For a multiple-phase system Θ , find an admissible trajectory that maximizes the objective function,

$$\Pi := \sum_{j=1}^n \left[\int_{t_{j-1}}^{t_j} F_j(x(t), u_j(t)) dt \right]$$

subject to

$$\begin{aligned} \dot{x} &= f_j(x(t), u_j(t)), t \in [t_{j-1}, t_j], \forall j \\ x(0) &= x_0; \\ x_k(t_n) &= x_{kf} \text{ for } k = \{1, \dots, K\} \text{ endpoint constraints; rest is free} \\ t_n &\text{ is free} \end{aligned}$$

Theorem B.2 (Doole 2005, thm. 2.1 - A Hybrid Maximum Principle) Consider a multiple-phase system Θ , as defined above,. Let $\{T^*, x^*(t), \{u_j^*(t)\}\}$ be an admissible trajectory, maximizing Π in the problem above. Call this optimal trajectory Γ^* .

Define a Hamiltonian function for each phase ("regime") as

$$H_j(x(t), u_j(t), \lambda_j(t), t) = F_j(x(t), u_j(t)) + \lambda_j(t) E_j(x(t), u_j(t)), \quad t \in [t_{j-1}, t_j]$$

An optimal trajectory Γ^* requires

1. Initial conditions x_0 to be satisfied
2. Vectors of real-valued, piecewise continuously differentiable adjoint functions $\lambda_j(t)$ that satisfy

$$\dot{\lambda}_j(t) = - \frac{\partial H_j(x(t), u_j(t), \lambda_j(t), t)}{\partial x(t)}$$

3. control function $u_j(t)$ for each phase satisfying

$$\frac{\partial H_j(x(t), u_j(t), \lambda_j(t), t)}{\partial u_j(t)} = 0$$

4. for any state variable k free at the end time t_n , when there is no 'scrap value' at the end point:

$$\lambda_{n,k}(t_n) = 0$$

5. for free end time t_n

$$H_n(x(t_n), u_j(t_n), \lambda_j(t_n), t_n) = 0$$

6. adjoint vectors that satisfy the phase-boundary conditions

$$\lambda_j(t_j) = \lambda_{j+1}(t_j),$$

at switching times t_1, \dots, t_{n-1} , and corresponding regimes $j = 0, \dots, n - 1$.

7. for those switching times that $t_{j-1} < t_j < t_{j+1}$

$$H_j(x(t), u_j(t), \lambda_j(t), t) \Big|_{t_j} = H_{j+1}(x(t), u_{j+1}(t), \lambda_{j+1}(t), t) \Big|_{t_j};$$

for those switching times such that, resp. $t_{j-1} = t_j < t_{j+1}, t_{j-1} < t_j = t_{j+1}$, we have

$$H_j(x(t), u_j(t), \lambda_j(t), t) \Big|_{t_j} \leq H_{j+1}(x(t), u_{j+1}(t), \lambda_{j+1}(t), t) \Big|_{t_j},$$

resp.

$$H_j(x(t), u_j(t), \lambda_j(t), t) \Big|_{t_j} \geq H_{j+1}(x(t), u_{j+1}(t), \lambda_{j+1}(t), t) \Big|_{t_j}.$$

Note that conditions (1.-5.) are the same as in a regular (single-phase) problem. This means also that we can split up the multiple-phase problem in parts. Conditions 6. and 7. link the parts, and are both forward looking²³. The co-state equations (6.) go backwards from the final time, so we can apply a form of backwards induction to find a candidate optimal solution (as proposition 4 also pointed at).

Let us first derive the necessary conditions for the problem at hand. Suppose we have productivities p_1, \dots, p_n , and a probability distribution over these productivities $h(p_i) = \alpha_i, i = 1, \dots, n$. Each phase j is associated with a different minimum productivity. The continuous control is $w_j(t)$. Moreover, from proposition 1, we know that the boundaries of the wage within that phase given by $p_{i-1} \leq w_j(t) \leq p_w$ are without loss of generality (of the problem at hand), and $p_0 = c$. Call \bar{v} the current value of the worker over unemployment, i.e. $\bar{v} \equiv \bar{V} - V^u$. Moreover, define v to be the expected ex ante net current value at a moment t : to be precise, it is defined as follows: if a worker expects to have a current value \bar{V} if he would have stayed in the match and the firm did not fire him. Suppose with probability α the firm would have fired him. Then

$$v \equiv (1 - \alpha)(\bar{V} - V^u).$$

Defining our state to be v instead of \bar{V} means that we have no jumps in the value of the state at a switching point, thus keeping the optimal control itself relatively uncomplicated.

Then hybrid optimal control problem for the firm is:

$$\max_{\Gamma} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sum_{j=i}^n \psi(t) \alpha_j [p_j - w] dt \quad (57)$$

with

$$\dot{\psi}_i(t) = -[\delta + \lambda(1 - F(V))]\psi_i(t) \quad (58)$$

$$\dot{v}_i(t) = (1 - H(p_i))([\delta + \lambda(1 - F(V))]V - w - \lambda \int_V \hat{V} dF(\hat{V})) \quad (59)$$

and we still use our old acquaintance $V = v/[1 - H(p_i)] + V^u$ to condense notation. Initial conditions are

$$\psi(0) = 1, v(0) = V_0 - V^u.$$

Defining the Hamiltonian for each stage, as in the theorem above, we have

$$\begin{aligned} H_i = & \sum_{j=i}^n \psi(t) \alpha_j [p_j - w] - x_\psi [\delta + \lambda(1 - F(V))]\psi_i(t) \\ & + x_v (1 - H(p_i))([\delta + \lambda(1 - F(V))]V - w - \lambda \int_V \hat{V} dF(\hat{V})) + \mu_{i,1} w - \mu_{i,2} \bar{w} \end{aligned} \quad (60)$$

Now, the hybrid maximum principle (following thm. B.2) tells us that necessary conditions for an optimal Γ are, among others the evolution of the value of the co-state variables according to:

$$\dot{x}_v = -x_v[\delta + \lambda(1 - F(V))] - x_\psi \frac{\lambda F'(V)}{[1 - H(p_i)]} \quad (61)$$

$$\dot{x}_\psi = -\sum_{j=i}^n \alpha_j [p_j - w] + x_\psi [\delta + \lambda(1 - F(V))] \quad (62)$$

Integrating the last equation by parts, and using the endpoint and switching point conditions, we find:

$$x_\psi(t)\psi(t) = \int_t^{t_i} \left(\psi(s) \sum_{j=i}^n \alpha_j [p_j - w(s)] \right) ds + x_\psi(t_i); \quad (63)$$

²³In fact, given that there is no discounting in this problem, the value of the current value hamiltonian equals zero at any time given an optimal trajectory. Thus equation 7. is satisfied straightforwardly in any optimum. See Leonard and Long 1992, p.298.

We can derive from this that the ex ante expected current profit for the firm is given by

$$\frac{x_\psi(t)}{1 - H(p_i)} = \tilde{\Pi}(t). \quad (64)$$

A similar exercise for x_v , for the last stage, yields:

$$\frac{x_v(t)}{\psi(t)} = \int_t^{t_n} \lambda F'(V(s)) \Pi(s) ds + \frac{x_v(t_n)}{\psi(t_n)}, \quad t \in [t_{n-1}, t_n] \quad (65)$$

We can derive $x_v(t_n)$ from the endpoint condition: $H_n(t_n) = 0$, while (since ψ is free, we have $x_\psi(t_n) = 0$,

$$\psi(t) \alpha_n (p_n - w) + x_v (1 - H(p_i)) \left([\delta + \lambda(1 - F(V))] V - w - \lambda \int_V \hat{V} dF(\hat{V}) \right) = 0;$$

but,

$$[\delta + \lambda(1 - F(V))] V - w - \lambda \int_V \hat{V} dF(\hat{V}) = p_n - w,$$

because after t_n $w = p_n$ forever. Combining the two lines above, and we find

$$\frac{x_v(t_n)}{\psi(t_n)} = -1, \quad (66)$$

so that

$$\frac{x_v(t)}{\psi(t)} = \int_t^{t_n} \lambda F'(V(s)) \Pi(s) ds - 1, \quad t \in [t_{n-1}, t_n] \quad (67)$$

Now, the optimality conditions on the evolution of the costate dictate that

$$\frac{x_v(t)}{\psi(t)} = \int_t^{t_j} \lambda F'(V(s)) \Pi(s) ds + \sum_{i=j+1}^n \int_{t_{i-1}}^{t_i} \lambda F'(V(s)) \Pi(s) ds - 1, \quad t \in [t_{j-1}, t_j]. \quad (68)$$

Both $x_v(t)$, $\psi(t)$, and their fraction, are continuous, and piecewise continuously differentiable between switching points.

What we want to show next is that $x_v(t)/\psi(t)$ equals $\dot{\Pi}(t)/\dot{V}(t)$, and the switching point condition on the co-state variable reduces to the switching point condition derived in proposition 4. For this, on one hand, we plug in the results of the propositions 1-3 in the hybrid problem (we know that any optimum should satisfy these conditions)²⁴. On the other hand, we can write the evolution of the ratio $\dot{\Pi}(t)/\dot{V}(t)$ differently. To do this, we will differentiate this ratio with respect to t , along the equilibrium path, and integrate over it again:

$$\frac{d(\dot{\Pi}(t)/\dot{V}(t))}{dt} \Big|_{\Pi^*(V)} = \frac{d(\dot{\Pi}(t)/\dot{V}(t))}{dV} \Big|_{\Pi^*(V)} \cdot \frac{dV}{dt}$$

The first term of this derivative is:

$$\begin{aligned} & d \left(\frac{[\delta + \lambda(1 - F(V))] \Pi^*(V) - (p_h - p_l)}{[\delta + \lambda(1 - F(V))] V - p_l - \lambda \int_V \hat{V} dF(\hat{V})} \right) / dV \\ &= \frac{\lambda F(V) \Pi^*}{\dot{V}} + \frac{[\delta + \lambda(1 - F(V))] \dot{\Pi}}{\dot{V}^2} - \frac{[\delta + \lambda(1 - F(V))] \dot{\Pi}}{\dot{V}^2} \\ &= \frac{\lambda F(V) \Pi^*}{\dot{V}}, \end{aligned} \quad (69)$$

²⁴We have effectively reduced the optimization to a so-called ‘autonomous switched system’ (mainly studied in the realms of engineering and robotics), where the controlled variables are the switching time, but there is no continuous control in between. The solution approach through the co-state variable that is often taken in this literature is similar to the approach I take here (see Egerstedt, Wardi and Delmotte 2003, for example).

where we have used that $d\Pi^*(V)/dV = \dot{\Pi}(V)/\dot{V}$. To complete it, see that

$$\frac{d\left(\dot{\Pi}(t)/\dot{V}(t)\right)}{dV} \Big|_{\Pi^*(V)} \cdot \frac{dV}{dt} = \frac{\lambda F(V)\Pi^*}{\dot{V}} \cdot \dot{V} = \lambda F(V)\Pi^*$$

Thus, the conditions concerning the switching time stated in the hybrid maximum principle are equivalent to the switching conditions derived in proposition 4.

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