VIRTUAL PATH LAYOUTS IN ATM NETWORKS

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Abstract. We study virtual path layouts in a very popular type of fast interconnection networks, namely ATM networks. One of the main problems in such networks is to construct path layouts that minimize the hop-number (i.e. the number of virtual paths between any two nodes) as a function of the edge congestion \( c \) (i.e. the number of virtual paths going through a link). In this paper we construct for any \( n \) vertex network \( H \) and any \( c \) a virtual path layout with hop-number \( O\left(\frac{\text{diam}(H) \log \Delta}{\log c}\right) \), where \( \text{diam}(H) \) is the diameter of the network \( H \) and \( \Delta \) is its maximum degree. Involving a general lower bound from [6] we see that these hop-numbers are optimal for bounded degree networks with the diameter \( O(\log n) \) for any congestion \( c \). In the case of unbounded degree networks (with the diameter \( O(\log n) \)) these hop-numbers are optimal for any \( c \geq \Delta \). For instance, this gives optimal hop-numbers for hypercube related networks. Moreover, we improve known results for paths and meshes and prove optimal hop-numbers for hypercubes.

Key words. ATM network, congestion, hop-number, virtual paths layout

AMS subject classifications. 68M10, 90B12

1. Introduction. Broadband Integrated Services Digital Network (B-ISDN) is a new paradigm in digital communication which integrates previous distinct networks (telephone, cable television, computer...) into a single digital network. The basic network transmission medium is a fiber-optic cable capable of transferring data at very high rates. The bottleneck caused by slow software-based switches is resolved by special purpose fast hardware. To utilize this, a new multiplexing and switching technology called ATM (Asynchronous Transfer Mode) was proposed (see e.g. [9]). Packet routing in ATM networks is based on relatively small fixed-size packets. Messages may be transmitted through arbitrarily long virtual paths. Packets are routed along these paths by maintaining a routing field whose subfields determine intermediate destinations of the packet, i.e. end-points of virtual paths on its way to final destination. One of the main problems in such networks is to construct path layout that minimizes the hop-number (i.e. the number of virtual paths between any two nodes) as a function of the edge congestion \( c \) (i.e. the number of virtual paths going through a link).

In this paper we construct for any \( n \) vertex network \( H \) and any \( c \) a virtual path layout with the hop-number \( O\left(\frac{\text{diam}(H) \log \Delta}{\log c}\right) \), where \( \text{diam}(H) \) is the diameter of the network \( H \) and \( \Delta \) is its maximum degree. Involving a general lower bound from [6] we see that these hop-numbers are optimal for bounded degree networks with the diameter \( O(\log n) \) for any congestion \( c \). In the case of unbounded degree networks (with the diameter \( O(\log n) \)) these hop-numbers are optimal for any \( c \geq \Delta \). For instance, this gives optimal hop-numbers for hypercube related networks. Moreover, we improve known results for paths and meshes of Kranakis et al. [6]. Finally, we prove optimal hop-numbers for hypercubes.

1.1. Model, notation and results. The basic model of ATM networks was introduced by Gerstel et al. in [1, 2, 4, 5] and further developed in [3, 6]. By a network we understand any graph \( G = (V_G, E_G) \). By \( \text{diam}(G) \) we denote the diameter of \( G \). Similarly, by \( d(x, y) \) we denote the distance of the vertices \( x \) and \( y \) in \( G \). Let \( \Delta \) denote...
the maximum degree of a graph. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs such that $|V_G| \leq |V_H|$. An embedding of $G$ in $H$ is a pair of mappings $(\phi, \psi)$ satisfying

$$\phi : V_G \rightarrow V_H$$

such that if $uv \in E_G$ then $\psi(uv)$ is a path between $\phi(u)$ and $\phi(v)$. Define the congestion of an edge $e \in E_H$ as $cg(\phi, \psi)(e) = |\{f \in E_G : e \in \psi(f)\}|$ and the congestion of $G$ in $H$ as $cg(G, H) = \max_{e \in E_H} \{cg(\phi, \psi)(e)\}$. Note that if there is no confusion we will omit the subscript $(\phi, \psi)$ in $cg(\phi, \psi)(e)$.

The paths $\{\psi(E_G)\}$ are called virtual paths. The virtual path problem can be mathematically defined as: Given a graph $H$ and a positive number $c$, among all graphs $G$ satisfying $|V_G| = |V_H|$ and $cg(G, H) \leq c$, find a graph $G_0$ of minimum diameter. The minimum diameter is called the hop-number and denoted by $Hop_H(c)$.

Gerstel and Zaks [5] studied virtual path layouts for paths, cycles and meshes. They assumed an additional requirement that virtual channels are the shortest paths in the networks. Kranakis et al. [6] dropped the requirement and proved several optimal results on the hop-number of paths and meshes for small $c$. Namely, if $P_n$ denotes an $n$-vertex path then $Hop_{P_n}(2) = \sqrt{2n} + o(1)$, $Hop_{P_n}(c) = \Theta(n^{1/c})$, when $c$ is a constant and $Hop_{P_n}(\log^2 n / \log \log n) = \Theta(\log n / \log \log n)$, and if $M_n$ denotes an $n \times n$ mesh then $Hop_{M_n}(c) = \Theta(\log n)$, for constant $c$.

We assume the same model as Kranakis et al. [6]. Our main result says that for any $c \geq 1$ and any $n$ vertex graph $H$ of bounded degree and diameter $O(\log n)$, $Hop_H(c) = \Theta(\log n / \log c)$. Several standard networks belong to this class of graphs, e.g. mesh of trees, Butterfly, cube-connected-cycles, binary de Bruijn or shuffle-exchange graph. If $c \geq \Delta$, then we construct optimal virtual path layouts for some unbounded degree networks, e.g. star, pancake and $k$-ary de Bruijn and Kautz graphs.

Further we improve the results of Kranakis et al. [6] in the following way:

(i) $Hop_{P_n}(c) = \Theta\left(\frac{\log n}{\log c}\right)$, if $c \geq \log^{1+\varepsilon} n$, for any fixed $\varepsilon > 0$.

(ii) If $H$ denotes 2-dimensional $n \times n$ mesh then $Hop_H(c) = \Theta\left(\frac{\log n}{\log c}\right)$, for $c \geq 2$.

Finally we show that for the $n$-dimensional hypercube $Q_n$

$$Hop_{Q_n}(c) = \begin{cases} 
\Theta\left(\frac{n}{\log n}\right) & \text{if } 2 \leq c \leq n, \\
\Theta\left(\frac{n}{\log c}\right) & \text{otherwise.}
\end{cases}$$

2. General Bounds. In this section we describe a general method for constructing virtual path layouts and then apply it to some standard networks. Since we deal with asymptotic results only, we can restrict ourselves to looking for the graph $G_0$ among trees.

**Proposition 2.1.** Let a graph $G_0$ minimize $diam(G)$ with respect to $cg(G, H) \leq c$. Then there exists a tree $T$ with $diam(T) \leq 2diam(G_0)$ and $cg(T, H) \leq c$.

Indeed, we can take $T$ as a breadth first search spanning tree of $H_0$. We will often apply the following useful lemma from [10].

**Lemma 2.2.** Let $G_1$, $G_2$ and $G_3$ be graphs with $|V_{G_1}| \leq |V_{G_2}| \leq |V_{G_3}|$. Consider an embedding of $G_1$ into $G_2$ and an embedding of $G_2$ into $G_3$. Then $cg(G_1, G_3) \leq cg(G_1, G_2) \cdot cg(G_2, G_3)$.

It is worth noting that an embedding of $G_1$ into $G_2$ followed by the embedding of $G_2$ into $G_3$ induces an embedding of $G_1$ into $G_3$. An image of an edge under this composed embedding is not necessarily a path, but a walk in general. It is easy to transform the walk into a path by omitting multiple edges and vertices.
Throughout, we will use a general lower bound of Kranakis et al. [6].

**Lemma 2.3.** For any $n$ vertex graph $H$ of maximum degree $\Delta$ and for any $c \geq 1$
\[ \text{Hop}_H(c) \geq \frac{\log n}{\log(c\Delta)} - 1. \]

By $R(k, n)$ we denote the complete $k$-ary $n$-level tree. Note that by a $k$-ary 1-level
and 0-ary $k$-level tree we mean a single vertex. We sometimes abbreviate this notation
in the case of $R(2, n)$ to $B_n$ (the complete binary tree) and $R(l, 2)$ to $S_l$ (the star on
$l + 1$ vertices).

Let $T$ be a rooted tree. Let $G$ be a rooted tree with $l \geq k$ leaves, where $l$ and $k$
are positive integers. By $G \ast kT$ we denote the graph constructed as follows: Take $k$
copies of $T$ and identify the roots of the copies of $T$ with $k$ distinct leaves of $G$.

For $d, i, n \geq 1$ and $k = d'$ define the tree $F(k; d, n)$ recursively as follows. If $i = 1$
set $F(k; d, n) = R(d, n)$. If $i \geq n$ set $F(k; d, n) = R(\frac{d + 1}{d - 1} - 1, 2)$. If
$1 < i < n$, set $F(k; d, n) = R(\frac{d + i - 1}{d - 1} - 1, 2) \ast kF(k; d, n - i)$. Note that
the number of vertices of $F(k; d, n)$ is $d + n - 1$. A picture of $F(4; 2, 6)$ is depicted in
Fig. 1 (the left tree).

**Lemma 2.4.** For $d, i, n \geq 1$ and $k = d'$, $\text{diam}(F(k; d, n)) = 2(a + \lceil \frac{i}{k} \rceil) = 2\lceil \frac{n}{k - 1} \rceil$
where $n - 1 = ai + r, 0 \leq r < i$.

**Proof.** For $n = 1, 2$, the Lemma is easy to observe. We assume that Lemma is
true for all $l < n$ and we prove it for $l = n$. For $i = 1$, $\text{diam}(R(d, n)) = 2(n - 1)$, the
Lemma is true. Similarly, for $i \geq n$, $\text{diam}(R(\frac{d + 1}{d - 1} - 1, 2)) = 2$. Now, let $1 < i < n$.

Hence $F(k; d, n) = R(\frac{d + i - 1}{d - 1} - 1, 2) \ast kF(k; d, n - i)$. Let $n - 1 = ai + r, 0 \leq r < i$.
Since $n - i \geq 1$ and since $n + i = (a - 1)i + r$, by induction, $\text{diam}(F(k; d, n - i)) = 2(a - 1 + \lceil \frac{i}{k} \rceil)$,
and $\text{diam}(R(\frac{d + i - 1}{d - 1} - 1, 2)) = 2$. It follows from the construction of $F(k; d, n)$ that
$\text{diam}(F(k; d, n)) = \text{diam}(F(k; d, n - i)) + 2 = 2(a + \lceil \frac{i}{k} \rceil) = 2\lceil \frac{n}{k - 1} \rceil$. \[\Box\]

**Lemma 2.5.** For $d, i, n \geq 1$ and $k = d'$, $\text{cg}(F(k; d, n), R(d, n)) \leq \frac{k - 1}{d - 1}$.

**Proof.** We proceed by induction on $n$. For $n = 1, 2$, the result is easy to see.
Assuming the result is true for $l \leq n - 1$, we prove it for $l = n$. For $i = 1$, the
statement of Lemma is true (we take as the mapping $(\phi, \psi)$ the natural isomorphism
$\phi : F(k; d, n) \rightarrow R(d, n)$, and $\psi$ the induced mapping by $\phi$). If $i \geq n$, let $\phi$
map the root of $R(\frac{d + 1}{d - 1} - 1, 2)$ on the root of $R(d, n)$ and other vertices, injectively in
an arbitrary way, on remaining vertices of $R(d, n)$. Further let $\psi$ map each edge $e$
of $R(\frac{d + 1}{d - 1} - 1, 2)$ on the unique paths in $R(d, n)$ connecting the images of the end-
vertices of $e$. It is a matter of routine to observe that the congestion of each edge
of $R(d, n)$ induced by $(\phi, \psi)$ is at most $\frac{k - 1}{d - 1}$ (the maximum congestion is achieved on
edges incident with the root of $R(d, n)$), thus $\text{cg}(R(\frac{d + 1}{d - 1} - 1, 2), R(d, n)) \leq \frac{k - 1}{d - 1}$. Now,
let $1 < i < n$. By definition $F(k; d, n) = R(\frac{d + i - 1}{d - 1} - 1, 2) \ast kF(k; d, n - i)$. It holds that
$R(d, n) = R(d, i + 1) \ast kR(d, n - i)$. Using induction we can embed $kF(k; d, n - i)$ into
$kR(d, n - i)$ (each $F(k; d, n - i)$ into some $R(d, n - i)$) with congestion $\leq \frac{k - 1}{d - 1}$ and thus
$\text{cg}(kF(k; d, n - i), kR(d, n - i)) \leq \frac{k - 1}{d - 1}$. Similarly, we obtain $\text{cg}(R(\frac{d + i - 1}{d - 1} - 1, 2), R(d, i + 1)) \leq \frac{k - 1}{d - 1}$. Obviously, these two embeddings together induce an embedding $(\phi, \psi)$ of
$F(k; d, n)$ into $R(d, n)$ with congestion at most $\frac{k - 1}{d - 1}$. \[\Box\]

**Remark 2.1.** The embedding $(\phi, \psi)$, constructed in the previous proof, has the
property that for each vertex $v \in V_{F(k; d, n)}$ there is at most one its neighbour
$u \in V_{F(k; d, n)}$ such that the unique $\phi(v) - \phi(u)$ path in $R(d, n)$ contains a vertex $x$
with $d(\phi(x), r) < d(\phi(v), r)$, where $r$ is the root of $R(d, n)$. This can be observed in
Fig. 1, where the embedding of $F(4; 2, 6)$ in $R(2, 6)$ is depicted.
Remark 2.1, its preimage is also a leaf of a subgraph of $T$ (since $T$ is a tree with maximum degree at most $\Delta$ always exists). For each $cg_1$, clearly $c < \frac{1}{2}$. Similarly, by Lemma 2.6, there is a tree $T'$ with $cg(T',H) \leq \frac{1}{2}$. Hence, $T'$ is indeed the required tree.

Let $\phi'$ be the restricted mapping $\phi$ to the graph $T'$. Obviously, $\phi'$ is a surjection as well. Let $|V_T| = |V_L|$. Since $\psi$ maps each edge $xy$ to the unique $\phi(x) - \phi(y)$ path in $L$ and since $L$ and $T$ are trees, the restricted mapping $\psi$ to the graph $T'$, say $\psi'$, is well defined.

Hence $(\phi', \psi')$ is an embedding of $T'$ in $T$ with $cg(\phi', \psi')(T', T) \leq cg(\phi, \psi)(G, L) \leq \frac{(\Delta - 1)^{i+1} - 1}{\Delta - 2}$. Since $G$ is a tree, to finish the proof it is sufficient to prove that $T'$ is connected, since then $diam(T') \leq diam(G) \leq 2\left\lceil \frac{diam(T)}{2} \right\rceil$. Let $x$ be the last vertex added to $T$ in the construction of $L$. Since $x$ is a leaf in $L$ and by Remark 2.1, it follows that $\phi^{-1}(x)$ is a leaf in $G$ as well. Thus after removing $x$ and $\phi^{-1}(x)$, the resulting graphs will be connected. We can continue removing vertices from $L$ and its preimages in $G$ (in the opposite way than they were added to $T$) to obtain $T$ and $T'$, respectively. Since in each step the deleted vertex is a leaf of a subgraph of $L$, by Remark 2.1, its preimage is also a leaf of a subgraph of $G$, and thus $T'$ is connected.

Theorem 2.7. For any graph $H$ of maximum degree $\Delta \geq 3$ and any given $c \geq 1$,

$$Hop_H(c) = O\left(\frac{diam(H) \log \Delta}{\log c}\right).$$

Proof. Let $T$ be a breadth first search spanning tree of $G$. Then $T$ has the maximum degree at most $\Delta \geq 3$ and $diam(T) \leq 2diam(H)$. Find $i$ such that $\frac{(\Delta - 1)^{i+1} - 1}{\Delta - 2} \leq c < \frac{(\Delta - 1)^{i+1} - 1}{\Delta - 2}$. Now, by Lemma 2.6, there is a tree $T'$ with $cg(T', H) \leq \frac{(\Delta - 1)^{i+1} - 1}{\Delta - 2}$. Clearly $cg(T', H) \leq \frac{(\Delta - 1)^{i+1} - 1}{\Delta - 2} \leq c$. Similarly, by Lemma 2.6,

$$Hop_H(c) \leq diam(T') \leq 2\left\lceil \frac{diam(T)}{i} \right\rceil \leq 2\left\lceil \frac{2diam(H) \log(\Delta - 1)}{\log(\Delta - 2)c + 1 - \log(\Delta - 1)} \right\rceil \leq 2\left\lceil \frac{diam(H) \log(\Delta - 1)}{\log c}\right\rceil = O\left(\frac{diam(H) \log \Delta}{\log c}\right).$$

Fig. 1 The embedding of $F(4; 2, 6)$ in $R(2, 6)$.
Theorem 2.7 together with Lemma 2.3 has two important consequences.

**Corollary 2.8.** Let $H$ be a graph of order $n$ with $\Delta = O(1)$ and $\text{diam}(H) \leq O(\log n)$. Then $\text{Hop}_H(c) = \Theta \left( \frac{\log n}{\log c} \right)$ for any $c$.

Note that several standard networks belong to this class of graphs, e.g. mesh of trees, Butterfly, cube-connected-cycles, binary de Bruijn or shuffle-exchange graph.

**Corollary 2.9.** Let $H$ be a graph of order $n$ with maximum degree $\Delta \geq 3$ satisfying $\text{diam}(H) \log \Delta = O(\log n)$. Then $\text{Hop}_H(c) = \Theta \left( \frac{\log n}{\log c} \right)$ for $c \geq \Delta$. If $c \geq \Delta$, then the above corollary gives optimal virtual path layouts for e.g. star, pancake and $k$-ary de Bruijn and Kautz graphs.

### 3. Paths and Meshes.

Let $P_n$ denote an $n$ vertex path and $M_n = P_n \times P_n$ denote an $n \times n$ mesh. The next theorem improves virtual path layouts for paths and meshes of Kranakis et al. [6].

**Theorem 3.1.** Let $c \geq 1$. Then

(i) $\text{Hop}_{P_n}(c) = \Theta \left( \frac{\log n}{\log c} \right), \text{ for } c \geq \log^{1+\varepsilon} n, \text{ for any fixed } \varepsilon > 0.$

(ii) $\text{Hop}_{M_n}(c) = \Theta \left( \frac{\log n}{\log c} \right), \text{ for } c \geq 2.$

**Proof.** Both lower bounds easily follow by applying Lemma 2.3. Consider the upper bounds. To prove (i), first find $m$ and $k = 2^i$ such that

$$2^m - 1 \leq n < 2^{m+1} - 1 \text{ and } (k - 1)(m + 2)/2 + 1 \leq c < (2k - 1)(m + 2)/2 + 1.$$ 

Setting $G_1 = F(k; 2, m), G_2 = B_m$ and $G_3 = P_{2^m - 1}$ in Lemma 2.2, using a result of Lengauer (based on the orthogonal projection of $B_m$ on a horizontal line) [8]:

$c g(B_m, P_{2^m - 1}) \leq (m + 2)/2$ and by Lemma 2.5 we get an embedding of $F(k; 2, m)$ into $P_{2^m - 1}$ with

$$c g(F(k; 2, m), P_{2^m - 1}) \leq (k - 1)(m + 2)/2.$$

Now choose $n - 2^m + 1$ new vertices and join them to leaves of $F(k; 2, m)$ (at most two per each leaf) in such a way that the resulting graph, say $T$, has $m + 1$ levels. Note that this is always possible. Clearly, $T$ has $n$ vertices. It is easy to extend the above embedding of $F(k; 2, m)$ into $P_{2^m - 1}$ to an embedding of $T$ into $P_n$ with

$$c g(T, P_n) \leq c g(F(k; 2, m), P_{2^m - 1}) + 1 \leq c.$$

Moreover, by Lemma 2.4

$$\text{Hop}_{P_n}(c) \leq \text{diam}(T) \leq \text{diam}(F(k; 2, m)) + 2 = 2 \left[ \frac{m - 1}{i} \right] + 2 \leq \frac{2m}{\log k} + 4 \leq \frac{2 \log(n + 1)}{\log c} + 4 \leq \frac{2 \log(n + 1)}{\log c + 1} + 4 = O \left( \frac{\log n}{\log c} \right),$$

for $c \geq \log^{1+\varepsilon} n$. 


For (ii) assume first the case \( n = 2^m, m \geq 1 \). Find \( k = 2^i \) such that
\[
2(k - 1) + 1 \leq \frac{c}{4} < 2(k - 1) + 1.
\]
Set \( G_1 = F(k; 2, 2m), G_2 = B_{2m}, \) and \( G_3 = M_n \) in Lemma 2.2. Zienicke [11] showed that \( cg(B_{2m}, M_n) = 2 \). Now we add one new vertex and edge to \( F(k; 2, 2m) \) resulting in a graph \( F \) on \( n \) vertices embeddable into \( M_n \) with \( cg(F, M_n) \leq 2(k - 1) + 1 \leq c/4 \). The diameter of \( F \) is increased by at most 1.

Second, consider an arbitrary \( n \). Find \( m \) such that \( 2^m - 1 \leq n < 2^{m+1} - 1 \). Embed the graph \( F \) into left lower submesh of \( M_n \) of size \( 2^m \times 2^m \). Similarly embed the graph \( F \) into other three “corner” submeshes of \( M_n \). We get a virtual path layout for \( M_n \) with congestion at most \( 4cg(F, M_n) \leq c \) and the hop-number
\[
\text{Hop}_{M_n}(c) \leq 2\text{diam}(F) \leq (2\text{diam}(F(k; 2, 2m)) + 1) \leq 2 \left\lceil \frac{2m-1}{i} \right\rceil + 2
\]
\[
\leq 2 \frac{\log(n+1)}{\log \left( \frac{c+2}{16} \right)} + 4 = O \left( \frac{\log n}{\log c} \right).
\]

Note that \( \text{Hop}_{P_n}(c) \) for non-constant \( c \leq \log n \) remains an open problem.

4. Hypercubes. The main result of this section is an optimal virtual path layout for hypercubes. The \( n \)-dimensional hypercube, denoted by \( Q_n \), is defined, by means of Cartesian product of graphs, as \( Q_0 = v \) (a single vertex) and \( Q_n = Q_{n-1} \times P_2 \). Edges of \( Q_n \) are divided into \( n \) groups in a natural way according to dimensions they belong to.

Let us define the tree \( T_1 \) to be a single vertex. Now, for \( n \geq 2 \) let the tree \( T_n = S_{2k-2} \ast kT_{n-i} \), where \( k = 2^i \leq n < 2^{i+1} \). Note that the number of vertices of \( T_n \) is \( 2^n - 1 \).

**Lemma 4.1.** The diameter of \( T_n \) satisfies
\[
\text{diam}(T_n) < \frac{8n}{\log n} + 2\lfloor \log n \rfloor.
\]

**Proof.** Let \( \text{depth}(T_n) \) denote the depth of \( T_n \). Then
\[
\text{depth}(T_n) = \text{depth}(T_{n-i}) + 1,
\]
where \( i = \lceil \log n \rceil \). Solving this recurrence we get
\[
\text{depth}(T_n) \leq \left\lceil \frac{n - 2^i + 1}{i} \right\rceil + \left\lceil \frac{2^i - 2^{i-1}}{i-1} \right\rceil + \ldots + \left\lceil \frac{2^2 - 2^1}{1} \right\rceil
\]
\[
\leq \sum_{j=1}^{i} \left\lceil \frac{2^j}{j} \right\rceil \leq \sum_{j=1}^{i} \frac{2^j}{j} + i \leq \frac{2^{i+1}}{i} + i \leq \frac{4n}{\log n} + \lfloor \log n \rfloor,
\]
where the estimation of the sum was done by a straightforward induction. Noting that \( \text{diam}(T_n) \leq 2\text{depth}(T_n) \) we have the result. \( \square \)

Let \( G \) be a graph and let \( S \) (\( U \)) be a subset of \( E_G \) (\( V_G \)). We define the graph induced by \( U \) under \( S \) as the graph with the vertex set \( U \) and all edges \( xy \in (U \times U) \cap S \).

Let \( S \) be the set of all edges in some \( i \leq n \) dimensions of \( Q_n \). Then the graph \( Q_n - S \) consists of \( 2^i \) copies of \( Q_{n-i} \), say \( Q^1, Q^2, \ldots, Q^{2^i} \). Let \( v^1 \) be a vertex of \( Q^1 \).
Let \( v^i \in Q^l \), \( 2 \leq l \leq 2^i \) be the corresponding vertices to \( v^1 \) i.e. \( v^l, 2 \leq l \leq i \) is the copy of \( v^1 \) chosen in the natural way. Then the graph induced by \( \{ v^1, v^2, \ldots, v^{2^i} \} \) under \( S \) is an \( i \)-dimensional hypercube, say \( \varphi,\psi \). Observe that the hypercubes \( Q_i(x) \) for all \( x \in Q^1 \) are vertex disjoint. Moreover, the mapping \( \xi : Q^l \to Q^l, l = 2, \ldots, 2^i \), given by \( \xi(x) = y \), where \( y \in Q^1 \cap Q_i(x) \), is the isomorphism induced by \( Q_i(v^i) \).

**Lemma 4.2.** For all \( n \geq 1 \), \( cg(T_n, Q_n) \leq 2 \).

**Proof.** In fact we prove, by induction on \( n \), the following stronger statement: There exists an embedding \( (\phi, \psi) \) of \( T_n \) into \( Q_n \) with \( cg(e) \leq 2 \), for all edges \( e \in Q_n \), satisfying the following two additional conditions. Let \( r \) be the root of \( T_n \) and \( v \in V_{Q_n} \setminus \phi(V_{T_n}) \).

(i) There exists a \( \phi(r) - v \) path \( P \) such that each edge of \( P \) has congestion at most 1 and just one neighbour of \( v \) is on \( P \).

(ii) For all neighbours \( x \) of \( v \) in \( Q_n \), \( cg(vx) = 0 \).

Let us call the vertex \( v \) a free vertex, the path \( P \) a free path and the embedding a good embedding. For \( n = 1, 2, 3 \) the statement is easy to observe. Thus we assume we have proved the statement for all integers up to \( n - 1 \) and we prove it for \( n \geq 4 \). By definition, \( T_n = S_{2k-2} * kT_{n-i} \), where \( k = 2^i \leq n \leq 2^{i+1} \). Let us delete all edges in some \( i \) dimensions of \( Q_n \) obtaining thereby \( k \) copies of \( Q_{n-i} \), say \( Q^1, \ldots, Q^k \) (see Fig. 2).

\[
(v^i_c,\phi) \circ (c^i,\psi) \circ (c^i,\psi) \\
(v^i_c,\phi) \circ (c^i,\psi) \circ (c^i,\psi) \\
(v^i_c,\phi) \circ (c^i,\psi) \circ (c^i,\psi)
\]

**Fig. 2** The \( k \) subhypercubes \( Q_{n-i} \)

For the notational convenience let us denote some important vertices of \( T_n \). By \( r \) we denote the root of \( T_n \) (this is also the root of \( S_{2k-2} \)). Further, by \( r^l \) we denote the leaf of \( S_{2k-2} \) which is the root of \( F^l = T_{n-i} \) \( (l = 1, 2, \ldots, k) \). Note that \( r^l \) are the vertices at which the subtrees \( T_{n-i} \) are amalgamated with \( S_{2k-2} \) in \( T_n \). Finally, for \( x^l \) \( (l = 3, 4, \ldots, k) \) we denote the \( k - 2 \) remaining leaves of \( S_{2k-2} \).

Let \( v^1 \) be any vertex of \( Q^1 \). In what follows we describe for \( l = 1, \ldots, k \) a good embedding of \( F^l \) into \( Q^l \). By the induction hypothesis there is a good embedding \( (\phi_1, \psi_1) \) of \( F^1 \) into \( Q^1 \). Since \( Q^1 \) is vertex transitive, we may assume that \( v^1 \) is the free vertex in \( Q^1 \). Moreover, there is a free \( \phi_1(r^1) - v^1 \) path, say \( P^1 \). By \( v^l \) we denote the vertex corresponding to \( v^1 \) in \( Q^l \). It follows from the symmetry of \( Q_{n-i} \) that there is a good embedding \( (\phi_2, \psi_2) \) of \( F^2 \) into \( Q^2 \) (obtained from the embedding of \( F^1 \) into \( Q^1 \) for which \( \phi_2(r^2) = v^2 \) and the free vertex in \( Q^2 \), say \( v \), is the vertex corresponding to \( \phi_1(r^1) \)). Furthermore, there is a \( v^2 - v \) free path \( P^2 \) in \( Q^2 \). Finally, for \( l = 3, \ldots, k \), by the existence of the isomorphism induced by \( Q_i(v^i) \), there is a good embedding \( (\phi_l, \psi_l) \) of \( F^l \) into \( Q^l \) for which \( \phi_l(r^l) \) is the vertex corresponding to \( \phi_1(r^1) \) and \( v^l \) is the free vertex in \( Q^l \). By the same argument, there is a \( v^l - \phi_l(r^l) \)
free path $P^l$ in $Q^l$, $l = 3, \ldots, k$.

Since $Q^1 \cup Q^2 \cup \ldots \cup Q^k$ is a factor of $Q_n$ and since $F_1 \cup F_2 \cup \ldots \cup F_k \cong kT_{n-1}$, the embeddings $(\phi_l, \psi_l)$, $l = 1, \ldots, k$, together induce an embedding $(\phi, \psi)$ of $kT_{n-1}$ into $Q_n \setminus \{v, v^1, v^2, v^3, \ldots, v^k\}$. To obtain the required embedding we put $\phi(r) = v^1$ and $\phi(x^l) = v^l$ for $l = 3, 4, \ldots, k$. Finally, we have to embed the edges of $S_{2k-2}$. Since all the vertices of $S_{2k-2}$ are already embedded, it is enough to describe the paths $v^1 - \phi(r^1)$, $l = 1, 2, \ldots, k$, and the paths $v^1 - v^l$, $l = 3, 4, \ldots, k$.

The path $v^1 - \phi(r^1) = P^1$ and the path $v^1 - \phi(r^2) = v^1v^2$. We construct the remaining paths using neighbours of the vertex $v^1$ in $Q^1$. Since the degree of $Q_n$ is $n \geq k$ and since $P_1$ is a free path, there are $k - 2$ neighbours of $v^1$ which lie neither on $P^1$ nor on $v^1v^2$, say $c_1^j, c_2^j, \ldots, c_k^j$. For each vertex $c^j$, $j = 3, 4, \ldots, k$, there is the hypercube $Q_i(c^j)$. Let $c^j$ be the corresponding vertex for $c^j$ in $Q^1$. Since the mapping on corresponding vertices is an isomorphism, the vertex $c^j$ is adjacent to the vertex $v^j$. Since $Q_i(c^j)$ is connected, there is a $c^j - c^j$ path, say $P(c^j, c^j)$, in $Q_i(c^j)$. Note that all $P(c^j, c^j)$ for $j = 3, 4, \ldots, k$ are vertex disjoint. Now, we are able to describe all remaining paths. For $j = 3, 4, \ldots, k$, the path $v^1 - v^j = (v^1c^j) \circ P(c^j, c^j) \circ (c^jv^j)$, and the path $v^1 - \phi(r^j) = (v^1c^j) \circ P(c^j, c^j) \circ (c^jv^j) \circ P^j$, where $\circ$ it the concatenation operation.

The embedding is now completely defined. It is easy, but time consuming exercise, to observe that $c \phi(e, v) \leq 2$, for each edge $e \in Q_n$. If we define $P = (v^1v^2) \circ P^2$, then since $P_1$ is a free path, $P$ is a $\phi(r) - v$ free path as well. Moreover, no edge incident with $v$ is used in the embedding, thus $c \phi(vx, v) = 0$ for all neighbours $x$ of $v$ and the embedding $(\phi, \psi)$ is good.

**Theorem 4.3.** For the $n$-dimensional hypercube we have

$$
\text{Hop}_{Q_n}(c) = \begin{cases} 
\Theta\left(\frac{n}{\log \epsilon}ight) & \text{if } 2 \leq c \leq n, \\
\Theta\left(\frac{n}{\log \epsilon}\right) & \text{otherwise.}
\end{cases}
$$

**Proof.** The upper bound of the first case follows from Lemmas 4.1 and 4.2. The lower bound is implied by Lemma 2.3. The second case is proved in a similar way as the result for the mesh using the fact from [7] that the complete binary tree on $2^n - 1$ vertices can be embedded in the $n$-dimensional hypercube with congestion 1.

In particular, the above result says that if the congestion $c$ is smaller than the degree of the hypercube then the hop-number does not depend on $c$. It is an interesting open question whether this holds for other important networks of unbounded degree like $k$-ary de Bruijn and star graph. Another open problem is to find an optimal virtual path layout for the $n$-vertex path if $c$ is non-constant and less than or equal to $\log^{1+\epsilon} n$, for $\epsilon > 0$.

**REFERENCES**


