The Extra-Factor Phenomenon Revisited: Unidimensional Unfolding as Quadratic Factor Analysis

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The application of linear factor analysis to a set of unfoldable (unidimensional) items produces a two-dimensional solution, called the extra-factor phenomenon, which potentially results in incorrect conclusions about the nature of a set of items (van Schuur & Kiers, 1994). Many explanations have been offered for this phenomenon. This study attempted further clarification within the general theory of factor analysis. Specifically, it was demonstrated that the extra-factor phenomenon arises because: (1) the metric unidimensional unfolding model is equivalent to the unidimensional quadratic factor model; and (2) at the level of covariance structure, the unidimensional quadratic factor model is not distinguishable from the two-dimensional linear factor model (McDonald, 1967). Also discussed are a number of theoretical linkages and bases of distinguishability that exist between unidimensional unfolding and linear factor analysis. *Index terms: equivalence of models, extra-factor phenomenon, linear factor analysis, quadratic factor analysis, unfolding analysis.*

Van Schuur & Kiers (1994) noted that the failure to detect a unidimensional unfolding structure will often result in incorrect conclusions. In particular, they noted that the linear factor analysis of a set of unfoldable (unidimensional) items produces a two-dimensional (2D) solution, which they referred to as the "extra-factor phenomenon." They considered a number of explanations for this phenomenon: "The problem arises because unfoldable data violate fundamental assumptions of the factor analysis model. Factor analysis assumes that values of the observed variables are linearly (or even monotonically) related to values on the underlying latent variables" (p. 97); "the extra-factor phenomenon is most dramatic when the unfolding representation is unidimensional, because the two-factor solution treats what in fact are the two halves of the unfolding dimension as independent" (p. 101); and "factor analysis is an inappropriate translation of the analyst's assumptions about the structure of a data set that conforms to the unidimensional unfolding model" (p. 99).

Applying linear factor analysis to a set of unfoldable items appears to result in inappropriate conclusions. In this paper, further clarification of the nature of the extra-factor phenomenon is obtained by phrasing it in terms of the general theory of factor analysis. The explanation rests on a demonstration of the equivalence of the metric unidimensional unfolding model to the quadratic factor model. This explanation possesses several attractive features. First, Ross & Cliff's (1964) original "extra factor" proof was stated in terms of principal component analysis. As noted by van Schuur & Kiers (1994), "The presence of unique factors in addition to common factors makes it even more problematic to reconstruct the unfolding representation from the factor solution" (p. 101). The explanation provided here is phrased entirely in terms of factor analysis, thus overcoming Ross and Cliff's limitation. Second, there is no doubt as to where the source of the extra-factor phenomenon lies: It is not with factor analysis, but with the inappropriate application of *linear* factor

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analysis. Finally, as a byproduct, this explanation illuminates interesting relationships and points of distinguishability that exist between the unidimensional unfolding and linear factor models.

Metric Unidimensional Unfolding and Quadratic Factor Analysis

In delineating the features of the quadratic factor models and unfolding models, distinctions must be made between fixed-score and random versions of these latent-variable models (e.g., Bartholomew, 1981; McDonald, 1979; van der Leeden, 1990). In a random latent-variable model, the latent—and manifest—variates are random variates. Therefore, they are represented by density functions. Conversely, in a fixed-score latent variable model, the latent variables are not random variates; instead, they are parameters to be estimated. These parameters are often referred to as *factor scores* or *person parameters.* Also, because their number increases with the sampling of persons, they are also incidental parameters (Neyman & Scott, 1948). In a fixed-score model, the factor scores can be estimated or treated as nuisance parameters that complicate the estimation of the item parameters. In a random model, the factors are random variates and are not estimated. Instead, interest is in the joint moments of the latent and manifest variates. Fixed-score and random versions of both the metric unidimensional unfolding model and the quadratic factor model can be derived.

Metric Unidimensional Unfolding Model

Model 1. A fixed-score metric unidimensional unfolding model (e.g., van Schuur & Kiers, 1994) for $i = 1, 2, \ldots, N$ persons measured on $j = 1, 2, \ldots, p$ items can be written as

$$z_{ij} = a_j + t_j(\theta_i - c_j)^2 + e_{ij} \ , \tag{1}$$

where
z_{ij} are $N \times p$ random (manifest) variates,
$E(e_{ij}) = 0 \ \forall \ i$ and j,
$V(e_{ij}) = \sigma^2_{ej} \ \forall \ i$ and j,
$f(e_{ij}, e_{lk}) = f_e(e_{ij}) f_e(e_{lk})$, unless $i = l$ and $j = k$,
$f_e(\cdot)$ and $f(\cdot)$ are density functions,
c_j is the location of item j,
θ_i is the location (ideal point) of person i on the latent continuum,
a_j is the expectation of z_{ij} when $\theta_i = c_j$,
t_j is a curvature parameter that determines the rate of change of the slope of $E(z_{ij})$, and
e_{ij} (a random variable) is a residual.
Because

$$E(z_{ij}) = a_j + t_j(\theta_i - c_j)^2 \ , \tag{2}$$

the mean response of person i to item j (over a population of hypothetical confrontations between the person and the item) is a function of the squared distance between θ_i and c_j. If t_j is negative, Equation 1 describes the inverted-U response function that is a characteristic of unfolding analysis. Davison (1977) discussed two close relatives of Model 1: a component version for infallible items and a classical test theory extension to the fallible case.

Model 2. A random version of Model 1 results from specifying a distribution for the ideal points. Assuming that the ideal points have a standard normal distribution,

$$z_j = a_j + t_j(\theta - c_j)^2 + e_j \ , \tag{3}$$

where
$\theta \sim \mathrm{N}(0, 1)$,
$E(e_j) = 0 \ \forall \ j$,
$V(e_j) = \sigma^2 e_j \ \forall \ j$,
$f(\boldsymbol{\theta}, e_j) = f_\theta(\boldsymbol{\theta}) f_e(e_j) \ \forall \ j$, and
$f(e_j, e_k) = f_e(e_j) f_e(e_k)$, unless $j = k$.

Unidimensional Quadratic Factor Model

Model 3. A fixed-score unidimensional quadratic factor model for N persons measured on p items (McDonald, 1967, 1979, 1983) can be written as

$$z_{ij} = \mu_j + \alpha_j \theta_i + \beta_j \theta_i^2 + e_{ij} \ , \tag{4}$$

where
μ_j is the intercept of j,
θ_i is the factor score for person i,
α_j is the regression coefficient for the linear component of j,
β_j is the regression coefficient for the quadratic component of j, and
e_{ij} are the uniquenesses, with the same distributions and covariances specified in Model 1.
To remove an indeterminacy in the scaling of the θ_i, $\sum \theta_i$ is set to 0 and $\sum \theta_i^2$ to 1.
Let
$\mathbf{Z}$ be an $N \times p$ matrix containing z_{ij};
$\boldsymbol{\mu}$, a $p \times 1$ vector of item intercepts containing μ_j;
$\mathbf{1}$, a $N \times 1$ vector of unities;
$\boldsymbol{\Phi}$, a $N \times 2$ matrix, with column 1 containing θ_i and column 2 containing θ_i^2;
$\boldsymbol{\Gamma}$, a $p \times 2$ matrix, with column 1 containing α_j and column 2 containing β_j; and
$\mathbf{E}$, a $N \times p$ matrix containing e_{ij}.
The model can then be written as

$$\mathbf{Z} = \mathbf{1}\boldsymbol{\mu}' + \boldsymbol{\Phi}\boldsymbol{\Gamma}' + \mathbf{E} \ . \tag{5}$$

Because $E(\mathbf{Z}) = \mathbf{1}\boldsymbol{\mu}' + \boldsymbol{\Phi}\boldsymbol{\Gamma}'$,

$$E(1/N)^*[\mathbf{Z} - E(\mathbf{Z})]'[\mathbf{Z} - E(\mathbf{Z})] = E[(1/N)^*\mathbf{E}'\mathbf{E}] = \boldsymbol{\Omega} \ , \tag{6}$$

where $\boldsymbol{\Omega}$ is a diagonal matrix containing σ_{ej}^2. Equation 6 is the covariance structure of Model 3. Equation 6 shows that, at the covariance structure level, Model 3 cannot be distinguished from a fixed-score 2D linear factor model (McDonald, 1983).

Model 4. A random version of Model 3 is generated by specifying a distribution for the factor scores. A common instantiation (e.g., McDonald, 1983) is

$$\mathbf{z} = \boldsymbol{\mu} + \mathbf{B}\mathbf{q} + \mathbf{e} \ , \tag{7}$$

where
$\mathbf{z}$ is a $p \times 1$ random vector of manifest variates,
$\mathbf{q} = [\boldsymbol{\theta}, 2^{-.5}(\boldsymbol{\theta}^2 - 1)]'$,
$\boldsymbol{\theta} \sim \mathrm{N}(0, 1)$,
$\mathbf{B}$ is a $p \times 2$ matrix containing the regression coefficients for the linear components of the items in column 1 and those for the quadratic components in column 2, and
$\mathbf{e}$ is a random vector of uniquenesses.

Note that the elements of $\mathbf{q}$ are normalized second and third Hermite-Tschebyscheff orthogonal polynomials in $\boldsymbol{\theta}$ (Kendall & Stuart, 1977, pp. 167–168). Hence, q_2 is a quadratic function of q_1; that is, $\boldsymbol{\theta}$, and $C(\mathbf{q}) = \mathbf{I}$.

Using the same distributional assumptions as in Model 2, the covariance structure of $\mathbf{z}$ can be expressed as

$$C(\mathbf{z}) = \mathbf{BB}' + \boldsymbol{\Omega} \; . \tag{8}$$

Equation 8 shows that, at the level of covariance structure, Model 4 is the same as the random 2D linear factor model (McDonald, 1983).

Equivalence of Models

Model 1 can be reparameterized as follows:

$$z_{ij} = a_j + t_j\theta_i^2 - 2t_j\theta_i c_j + t_j c_j^2 + e_{ij} = w_j + \alpha_j\theta_i + \beta_j\theta_i^2 + e_{ij} \; , \tag{9}$$

where
$w_j = a_j + t_j c_j^2$,
$\alpha_j = -2t_j c_j$, and
$\beta_j = t_j$.
θ_i can be scaled so that $\sum\theta_i = 0$ and $\sum\theta_i^2 = 1$. Let $\mathbf{w}$ be a $1 \times p$ vector containing w_j. Model 1 can then be rewritten as

$$\mathbf{Z} = \mathbf{1w}' + \boldsymbol{\Phi\Gamma}' + \mathbf{E} \; . \tag{10}$$

Thus, Model 1 is equivalent to Model 3.

Model 2 can also be reparameterized:

$$z_j = a_j + t_j\boldsymbol{\theta}^2 - 2t_j\boldsymbol{\theta}c_j + t_j c_j^2 + \mathbf{e}_j = w_j + \alpha_j\theta + \beta_j\theta^2 + e_j \; . \tag{11}$$

Let μ_j stand for $E(z_j) = w_j + \beta_j$. Then,

$$z_j = \mu_j + \alpha_j\theta + \beta(\theta^2 - 1) + e_j \; . \tag{12}$$

If $b_{j1} = \alpha_j$, $b_{j2} = 2^{.5}\beta$, and $q_2(\boldsymbol{\theta}) = 2^{-.5}(\boldsymbol{\theta}^2 - 1)$, then

$$z_j = \mu_j + b_{j1}\theta + b_{j2}q_2(\theta) + e_j \; . \tag{13}$$

Equation 13 can be written as

$$\mathbf{z} = \boldsymbol{\mu} + \mathbf{Bq} + \mathbf{e} \; , \tag{14}$$

where $B = [b_1 \;\; b_2]$. Therefore, Model 2 is equivalent to Model 4.

The Extra-Factor Phenomenon

Models 2 and 4 (1 and 3) are equivalent and, at the level of covariance structure, Model 4 (3) is not distinguishable from a 2D *linear* counterpart. This is the extra-factor phenomenon. In other words, the extra factor is what McDonald (1967) called a "spurious factor" due to the nonlinearity of item/factor regressions. The phenomenon is analogous to the classical "difficulty factor problem" that arises in the linear factor analysis of a set of dichotomous items (see McDonald & Ahlawat, 1974).

The item/factor regression that characterizes the unfolding model is second-degree polynomial. However, it is not possible (at least at the level of covariance structure) to distinguish between situations in which the items have (1) second-degree polynomial regressions on a single factor, and (2) linear regressions on *two* factors. Therefore, an apparently useful 2D linear factor solution could be a unidimensional unfolding structure in disguise (for empirical examples, see McDonald, 1967).

Bases of Distinguishability

The criterion by which factor analysts typically determine whether a set of items conforms to the 2D linear factor model (i.e., a particular covariance structure) does not rule out the possibility that the items instead conform to the unidimensional unfolding model. Thus, it is important to determine whether there are criteria that *do* distinguish between situations in which: (1) items conform to Model 1 or 2, and (2) items conform to the 2D linear factor model. The most comprehensive approach might be application of McDonald's (e.g., 1983) confirmatory nonlinear factor analysis, which allows hypotheses to be tested for the conformity of a set of items to a wide range of linear and nonlinear factor models. The decision as to whether either of the two above situations holds is a decision about the degree of polynomial that characterizes the item/factor regressions, given that a unidimensional factor model describes the items.

Simple, manifest grounds that distinguish between the two situations can be considered. The majority of results that appear to bear on the issue (e.g., Ross & Cliff, 1964; Davison, 1977) were derived for versions of Models 1 and 2 in which the items are viewed as error-free indicators of $\boldsymbol{\theta}$. Consequently, these results typically have been phrased in terms of the component structure of such unfoldable items, rather than of their factor structure. The question, then, is: On what bases can it be determined that a set of items conforms to Models 1 or 2 through linear factor analysis? Four possible bases of distinguishability are factor score distribution, covariance and correlation matrix, factor loadings, and partial correlation.

Factor Score Distribution

Although not distinguishable at the covariance structure level, the unidimensional unfolding model and the 2D linear factor model are distinguishable at the factor score distribution level, because the unfolding model is a unidimensional quadratic factor model (McDonald, 1967, 1983). If a set of items is described by Model 1 or 2, then this is detectable in the joint distribution of the factors from a 2D linear factor analysis of the items. Furthermore, quantitative features of this distribution can be employed to estimate the parameters of the unfolding model. Here, the argument is described for Model 2, although an analogous argument holds for Model 1.

Consider P manifest random variates (items), $\mathbf{z}$, that conform to Model 2, and let the corresponding 2D linear factor representation be

$$C(\mathbf{z}) = \mathbf{AA}' + \boldsymbol{\Omega} , \tag{15}$$

with random common factors $[\boldsymbol{\theta}_1^* \boldsymbol{\theta}_2^*]$. Because the items conform to Model 2, they also conform to Model 4. Therefore, $[\boldsymbol{\theta}_1^* \boldsymbol{\theta}_2^*]$ lie on a parabola. This can be used as a criterion for distinguishing between Situations 1 and 2 and as a basis for the estimation of the parameters of Model 2.

Specifically,

$$\theta_2^* = f\theta_1^{*2} + g\theta_1^* + h \tag{16}$$

for some set of constants f, g, and h. Because

$$\theta_1^* \sim \text{N}(0, 1)\ , \tag{17}$$

$$E(\theta_2^*) = E(f\theta_1^{*2} + g\theta_1^* + h) = 0\ , \tag{18}$$

$$V(\theta_2^*) = V(f\theta_1^{*2} + g\theta_1^* + h) = 1\ , \tag{19}$$

and

$$E(\theta_1^*\theta_2^*) = E(f\theta_1^{*3} + g\theta_1^{*2} + h\theta_1^*) = 0\ , \tag{20}$$

the constants f, g, and h can be determined. From Equations 17–20,

$$0 = E(\theta_1^*\theta_2^*) = fE(\theta_1^{*3}) + gE(\theta_1^{*2}) + hE(\theta_1^*) = g\ , \tag{21}$$

$$0 = E(\theta_2^*) = fE(\theta_1^{*2}) + gE(\theta_1^*) + E(h) = f + h\ , \tag{22}$$

and

$$1 = V(\theta_2^*) = E(f\theta_1^{*2} + g\theta_1^* + h)^2 - (f + h)^2 = 3f^2\sigma^4 + g^2 - f^2 = f^2(3\sigma^4 - 1)\ . \tag{23}$$

Therefore,

$g = 0$,
$h = -f$,
$f^2 = 1/2$, and
$\boldsymbol{\theta}_2^* = 2^{-.5}(\boldsymbol{\theta}_1^{*2} - 1)$.

From Equations 8 and 15, the regression coefficients of the quadratic factor representation are an orthogonal transformation of the regression coefficients of the 2D linear counterpart (McDonald, 1967). That is,

$$\mathbf{B} = \mathbf{AT}\ , \tag{24}$$

where $\mathbf{T}'\mathbf{T} = \mathbf{TT}' = \mathbf{I}$. From Equations 7 and 24, $\mathbf{T}'[\boldsymbol{\theta}_1^* \quad \boldsymbol{\theta}_2^*]' = \mathbf{q}$. $\mathbf{T} = \mathbf{EL}$, where $\mathbf{E}$ is an orthonormal reflection matrix, and $\mathbf{L}$ is an orthogonal rotation matrix. $\mathbf{E}$ can effect a reflection of c_j. The absolute magnitudes of c_j are invariant under any $\mathbf{E}$; therefore, any $\mathbf{E}$ can be selected. Let $\mathbf{z}_B$ contain a pair of Bartlett variates (factor predictors; Mardia, Kent, & Bibby, 1982), i.e.,

$$\mathbf{z}_B = (\mathbf{A}'\boldsymbol{\Omega}^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Omega}^{-1}(\mathbf{z} - \mathbf{u})\ . \tag{25}$$

Then, from Equations 14, 24, and 25,

$$\begin{aligned}\mathbf{z}_B &= (\mathbf{A}'\boldsymbol{\Omega}^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Omega}^{-1}(\mathbf{z} - \mathbf{u}) = (\mathbf{A}'\boldsymbol{\Omega}^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Omega}^{-1}(\mathbf{Bq} + \mathbf{e}) \\ &= (\mathbf{A}'\boldsymbol{\Omega}^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Omega}^{-1}(\mathbf{ATq} + \mathbf{e}) \\ &= (\mathbf{A}'\boldsymbol{\Omega}^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Omega}^{-1}\mathbf{ATq} + (\mathbf{A}'\boldsymbol{\Omega}^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Omega}^{-1}\mathbf{e} = \mathbf{Tq} + \mathbf{e}^*\ .\end{aligned} \tag{26}$$

That is, the Bartlett variates are a perturbed orthogonal transformation of $\mathbf{q}$.

If σ^2_{ej} are small, the parabola will be detectable and the rotation $\mathbf{T}$ determinable from a visual inspection of the joint distribution of the two Bartlett variates. McDonald (1967) found an analytic solution for $\mathbf{T}$ in terms of the higher-order moments of $\mathbf{Tq}$ and $\mathbf{e}^*$. After $\mathbf{T}$ is determined,

$$\boldsymbol{\alpha} = \mathbf{b}_1 = \mathbf{At}_1 \ , \tag{27}$$

and

$$\boldsymbol{\beta} = (1/2^{.5})^* \mathbf{b}_2 = (1/2^{.5})^* \mathbf{At}_2 \tag{28}$$

(from Equations 12 and 13). Then, from Equation 9, the unfolding parameters are recovered as

$$t_j = \beta_j \tag{29}$$

and

$$c_j = -\alpha_j/2t_j = -\alpha_j/2\beta_j \ . \tag{30}$$

As previously noted, c_j are, in (unfolding) Model 2, the locations of the items on the latent continuum. In the equivalent quadratic factor phrasing of Model 2, they are similarly the points at which the regression functions,

$$\mathbf{E}(\mathbf{z}_j|\theta) = \boldsymbol{\mu}_j + b_{j1}\theta + b_{j2}h_2(\theta) \ , \tag{31}$$

are at their maxima, because

$$\frac{dE(z_j|\theta)}{d\theta} = b_{j1} + \sqrt{2}b_{j2}\theta \ , \tag{32}$$

and

$$\theta_{\max} = -b_{j1}2^{-.5}/b_{j2} = -\alpha_j/2t_j = c_j \ . \tag{33}$$

t_j, on the other hand, are curvature parameters—they determine the rates of change of the slopes of the regression functions. Because each item/factor regression is a parabola, this rate of change is constant, $2t_j$.

This reasoning is easily generalized to multidimensional versions of Models 1 and 2. Specifically, items conforming to a t-dimensional generalization of Model 2 will have a $2t$-dimensional linear factor representation. In theory, an unfolding structure could be detected and its parameters estimated through linear factor analysis. In this case, however, a $2t$-dimensional distribution of Bartlett variates must be considered, and the t distinct factors must be matched to their quadratic components. Ross & Cliff (1964) established that, if the items are infallible (i.e., conform to a t-dimensional version of Model 2 in which there are no uniquenesses), their component representation would be of dimensionality $(t + 1)$.

Covariance and Correlation Matrix

Davison (1977) established that the correlation matrix of an unfoldable item set manifests a characteristic simplex pattern when the columns and rows of the matrix are ordered with respect to c_j. This fact can be used to differentiate between Situations 1 and 2. However, this result must be considered carefully because, once again, it applies to the case in which the items are infallible—an assumption that might be untenable. The question is whether Davison's result holds for items that conform to Model 1 or 2—models that are true factor models. The current results are based, without loss of generality, on an analysis of Model 2.

Let $\mathbf{\Sigma}$ be the covariance matrix of a set of items conforming to Model 2. Let the 2D linear factor representation of the items be as in Equation 15, and let $\mathbf{D}^{-.5}$ be the diagonal matrix containing the reciprocals of the standard deviations, σ_j, of the items. Then, the correlation matrix of the items is

$$\mathbf{R} = \mathbf{D}^{-.5}\mathbf{\Sigma}\mathbf{D}^{-.5} = \mathbf{D}^{-.5}\mathbf{A}\mathbf{A}'\mathbf{D}^{-.5} + \mathbf{D}^{-.5}\mathbf{\Omega}\mathbf{D}^{-.5} , \tag{34}$$

which, by Equation 24, can be re-expressed as

$$\mathbf{D}^{-.5}\mathbf{B}\mathbf{T}'\mathbf{T}\mathbf{B}'\mathbf{D}^{-.5} + \mathbf{D}^{-.5}\mathbf{\Omega}\mathbf{D}^{-.5} = \mathbf{F}\mathbf{F}' + \mathbf{\Omega}^* . \tag{35}$$

$\mathbf{F}$ is an orthogonal transformation of matrix $\mathbf{G} = \mathbf{D}^{-.5}\mathbf{B}$, which contains elements

$$g_{j1} = \frac{\alpha_j}{\sigma_j} = \frac{-2t_j c_j}{\sigma_j} \quad \text{and} \quad g_{j2} = \frac{\sqrt{2} \times \beta_j}{\sigma_j} = \frac{\sqrt{2} \times t_j}{\sigma_j} . \tag{36}$$

It then follows that ρ_{jk} (the correlation between items j and k) is equal to

$$g_{j1} \;\; g_{k1} + g_{j2} \;\; g_{k2} = \frac{2t_j t_k (2c_j c_k + 1)}{\sigma_j \sigma_k} , \tag{37}$$

where

$$\sigma_i = \left[2t_i^2(2c_i^2 + 1) + \sigma_{ei}^2\right]^{1/2} . \tag{38}$$

A simplex will not necessarily be manifested by the correlation matrix $\mathbf{R}$ that conforms to Model 2, because ρ_{jk} vary as a function of three distinct parameters, σ_{ek}, c_k, and t_k, within row j of $\mathbf{R}$. If, however, both t_i and the standard deviations, σ_i, are approximately equal across items, then the simplex pattern will be manifest in $\mathbf{R}$ when the rows and columns are ordered with respect to c_i (as by $\mathbf{\Sigma}$ when t_i are approximately equal).

Factor Loadings

Davison (1977) found that the "loadings" from the "factor analysis" of a set of unfoldable items lie on a semicircle in the 2D "factor space." However, his analysis was based on error-free items. Furthermore, he did not consider the factor analytic representation of the items but, instead, their component representation. His result should then be: the component weights (rows of the eigenvectors) of items conforming to error-free counterparts of Model 1 or 2 lie on a semicircle in $\mathbb{R}^2$. The question, then, is: Can it be correctly determined whether an item set conforms to Models 1 and 2 by examining a plot of the factor loadings from the 2D linear factor analysis of $\mathbf{R}$? Specifically, it must be found whether these loadings lie on a semicircle in $\mathbb{R}^2$. The length of the vector from the origin to the point with coordinates $\{g_{j1}, g_{j2}\}$ (i.e., the factor loadings of item j) is equal to

$$\frac{\sqrt{2}t_j\sqrt{2c_j^2 + 1}}{\sigma_j} . \tag{39}$$

The length is therefore free to vary over items. Consequently, the loadings will not, in general, lie on a semicircle. On the other hand, the angle between the vector from the origin to the point with coordinates $\{g_{j1}, g_{j2}\}$ and the horizontal axis is

$$\phi_{gj,e1} = \tan^{-1}\langle -\sqrt{2}c_j \rangle \ . \tag{40}$$

As c_j ranges from $-\infty$ to $+\infty$, $\phi_{gj,e1}$ will range from $-\Pi/2$ to $+\Pi/2$. Consider two orderings of the items, one in terms of $\phi_{gj,e1}$ (from smallest to largest) and one in terms of c_j (also from smallest to largest). Because $\phi_{gj,e1}$ is a monotone increasing function of c_j, these orderings will agree. Hence, the numerical order of the angles between the plotted loadings and the horizontal axis will correspond to the numerical order of c_j, a result that is in agreement with Davison (1977). Of course, at the manifest level, this fact in no way helps distinguish between the two situations.

Partial Correlation

Davison (1977) established that $\rho_{jl.k}$, the partial correlation between items j and l, given item k, is negative when c_k is intermediate to c_j and c_l, and positive otherwise. This result does not necessarily hold if the items are fallible, as they are in Models 1 and 2. The numerator of $\rho_{jl.k}$ is

$$\sigma_{j1} - \frac{\sigma_{jk}\sigma_{lk}}{\sigma_k^2} = \mathbf{b}_j'\mathbf{b}_l - \frac{\mathbf{b}_j'\mathbf{b}_k\mathbf{b}_l'\mathbf{b}_k}{(\mathbf{b}_k'\mathbf{b}_k + \sigma_{ek}^2)} \ , \tag{41}$$

where b_i is the ith row of $\mathbf{B}$. Now, if $\sigma_{ei}^2 = 0 \ \forall \ i$—that is, if the items are infallible, Davison's result follows; under this condition, Equation 41 becomes

$$\mathbf{b}_j'\mathbf{b}_l - \frac{\mathbf{b}_j'\mathbf{b}_k\mathbf{b}_l'\mathbf{b}_k}{\mathbf{b}_k'\mathbf{b}_k} = \mathbf{b}_j'(\mathbf{I} - \mathbf{P}_k)\mathbf{b}_l = \mathbf{b}_j'(\mathbf{I} - \mathbf{P}_k)(\mathbf{I} - \mathbf{P}_k)\mathbf{b}_l = \mathbf{b}_j^{*\prime}\mathbf{b}_l^* \ , \tag{42}$$

where $P_k = \mathbf{b}_k(\mathbf{b}_k'\mathbf{b}_k)^{-1}\mathbf{b}_k'$. If $\sigma_{ei}^2 = 0 \ \forall \ i$, then $\sigma_{ek}^2 = 0$, and the numerator of $\rho_{jl.k}$ will equal the inner product of the orthogonal projections of $\mathbf{b}_j$ and $\mathbf{b}_l$ onto the vector $\mathbf{b}_k^{\perp}$ in $\mathbb{R}^2$ that is orthogonal to $\mathbf{b}_k$, namely, $\mathbf{b}_j^{*\prime}\mathbf{b}_l^*$. The angular deviations of $\mathbf{b}_j$ and $\mathbf{b}_l$ from the horizontal axis, and hence, $\mathbf{b}_k$, are a monotone function of c_i (i.e., Equation 37 also holds for $\mathbf{b}_i$). If $c_j = c_k = c_l$, then $\mathbf{b}_j^{*\prime} = \mathbf{b}_l^{*\prime} = [0 \ \ 0]$, and $\mathbf{b}_j^{*\prime}\mathbf{b}_l^* = 0$. If, however, c_k is intermediate to c_j and c_l, the components of $\mathbf{b}_j^*$ will be opposite in sign to those of $\mathbf{b}_l^*$, and $\mathbf{b}_j^{*\prime}\mathbf{b}_l^*$ and $\rho_{jl.k}$ will be negative in sign. If c_k is not intermediate to c_j and c_l, the components of $\mathbf{b}_j^*$ will be of the same sign as those of $\mathbf{b}_l^*$ and $\mathbf{b}_j^{*\prime}\mathbf{b}_l^*$, and $\rho_{jl.k}$ will be positive in sign. On the other hand, for Models 1 and 2, in which σ_{ei}^2 are not necessarily equal to zero, this result does not hold. This is clear because, as σ_{ek}^2 become large, the numerator of $\rho_{jl.k}$ approaches the value

$$\mathbf{b}_j'\mathbf{b}_l = 2t_jt_l(2c_jc_l + 1) \ , \tag{43}$$

a quantity whose sign does not depend on the numerical ordering of c_j, c_k, and c_l.

Example

Consider an example with five variates $\mathbf{z}$, constructed according to Equation 3 so as to conform to Model 2. The following parameter values were used:

$\mathbf{a} = \{5 \ \ 4 \ \ 2 \ \ 1 \ \ 0\}$,

$t = \{-1 \quad -1 \quad -1 \quad -1 \quad -1\}$, and
$c = \{-.674 \quad -.125 \quad 0 \quad .270 \quad .674\}$.
In addition, $[\boldsymbol{\theta} \quad \mathbf{e}]' \sim \mathrm{N}_6(\mathbf{0}, \boldsymbol{\Sigma})$, with $\boldsymbol{\Sigma}$, a diagonal matrix containing $[1 \quad .005 \quad .015 \quad .010 \quad .076 \quad .013]$. A total of 464 cases were randomly generated. The sample covariance matrix, **S**, of z_i was

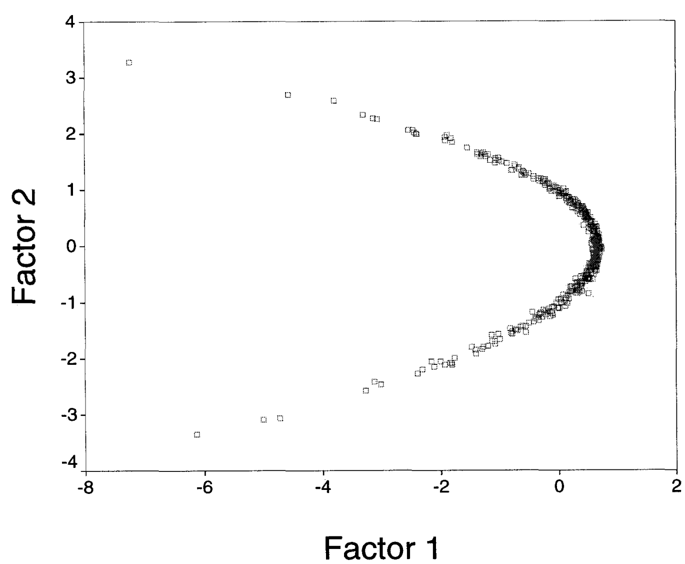
$$\mathbf{S} = \begin{pmatrix} 3.386 & 2.061 & 1.731 & 1.068 & .037 \\ 2.061 & 1.932 & 1.882 & 1.785 & 1.657 \\ 1.731 & 1.882 & 1.932 & 1.969 & 2.075 \\ 1.068 & 1.785 & 1.969 & 2.372 & 2.834 \\ .037 & 1.657 & 2.075 & 2.834 & 4.080 \end{pmatrix}, \tag{44}$$

which manifests a classical simplex pattern, a result of identical t_i and the rows and columns of **S** ordered in agreement with c_i. The eigenvalues of **S** were 9.69 3.93 .06 .02 .01. A 2D linear factor analysis of **S** (principal axis method) yielded the following estimate of **A**, the matrix of factor loadings:

$$\mathbf{A} = \begin{pmatrix} 1.185 & 1.402 \\ 1.344 & .335 \\ 1.384 & .063 \\ 1.443 & -.459 \\ 1.553 & -1.285 \end{pmatrix}. \tag{45}$$

If attention were paid only to the covariance structure of the items, the decision would be that the items conformed to the 2D linear factor model. However, evidence can easily be found that would result in the correct decision that the items have an unfolding structure. Figure 1 is a scatterplot of the two Bartlett variates. Clearly, the Bartlett variates lie on a parabola that has been rotated

Figure 1
Scatterplot of Bartlett Variates



approximately 275° counterclockwise from the x axis, a fact that can be recognized visually because of the uniformly small residual variances of the items. **L**, **E**, and **B** = **AT** were then estimated to be

$$\mathbf{L} = \begin{pmatrix} .087 & .996 \\ -.996 & .087 \end{pmatrix} , \tag{46}$$

$$\mathbf{E} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{47}$$

and

$$\mathbf{B} = \begin{pmatrix} -1.293 & -1.302 \\ -.217 & -1.368 \\ .058 & -1.384 \\ .583 & -1.397 \\ 1.415 & -1.435 \end{pmatrix} . \tag{48}$$

The first column of **B** is an estimate of $\boldsymbol{\alpha}$, and the second is an estimate of $2^{.5}\mathbf{t}$. The estimates for t_j were [−.921 − .967 − .979 − .988 − 1.015], and for $c_j = -\alpha_j/(2t_j)$ were [−.702 −.112 .030 .295 .697]. Although, in this case, **t** and **c** were well recovered, the results would not necessarily have been as clear if the residual variances had been large and the sample size small.

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