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An Analysis of Meehl's MAXCOV-HITMAX Procedure for the Case of Continuous Indicators

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MAXCOV-HITMAX was invented by Paul Meehl as a tool for the detection of latent taxonic structures (i.e., structures in which the latent variable, $\boldsymbol{\theta}$, is not continuously, but rather Bernoulli, distributed). It involves the examination of the shape of a certain conditional covariance function and is based on Meehl's claims that (R1) Taxonic structures produce single-peaked conditional covariance functions and that (R2) continuous latent structures produce flat, rather than single-peaked, curves. For neither (R1), nor (R2), have formal proofs been provided, Meehl and colleagues instead having provided an argument ("Meehl's Hypothesis") as to why they should be true, and a number of Monte Carlo studies. In an earlier article, Maraun, Slaney, and Goddyn (2003) proved that, for the case of dichotomous indicators, Meehl's Hypothesis is false and, by counterexample, that (R2) is false. In the current article (a) it is proved that, for the case of continuous indicators, Meehl's Hypothesis is false and (b) results are developed analytically on the behaviour of the conditional covariance functions produced by taxonic structures.

In a series of articles (Meehl, 1965, 1973, 1992; Meehl & Golden, 1982; Meehl & Yonce, 1996; Waller & Meehl, 1998), theoretician Paul Meehl developed what he calls *taxometrics*, a set of procedures that, he claims, may be used to detect latent taxa (i.e., discrete *types* which underlie, perhaps causally, responding to a set of indicator variables) when, in fact, such latent taxa exist. One of the most widely employed of these procedures is MAXCOV-HITMAX (hereafter, MAXCOV), which involves the examination of the shape of the covariance function of two indicator variates conditional on a third. The MAXCOV proce-

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dure was derived from Meehl's reasoning that (R1) Taxonic latent structures (hereafter, T-structures) should produce single-peaked conditional covariance functions and that (R2) continuous latent structures (hereafter, C-structures) should produce flat, rather than single-peaked, conditional covariance functions. If this were the case, MAXCOV could be employed to detect latent taxa, for it then could not only be used to judge when data were not in keeping with the hypothesis that they arose from a T-structure, but also to rule out the possibility that they arose from a C-structure.

However, there has recently arisen controversy regarding MAXCOV. Miller (1996) provided a counterexample that appeared to contradict (R2), suggesting that MAXCOV could signal taxonicity when the latent structure was, in fact, a C-structure. Maraun, Slaney, and Goddyn (2003) argued that Miller's counterexample was not relevant to a consideration of MAXCOV, because it featured a nonlinear component model rather than a structure from the (relevant) class of unidimensional monotone latent variable (UMLV) models. Questions have also been raised regarding the appropriateness of the popular practice of employing dichotomous indicators as input into MAXCOV. Maraun et al. reviewed criteria (necessary conditions) of T-structures for the case of dichotomous indicators, and proved that the reasoning that Meehl has offered in support of (R1) for the case of continuous indicators, does not hold for the case of dichotomous indicators. They also showed, by counterexample, that, for dichotomous indicators, (R2) is false. It should be noted that Meehl (1995) himself has claimed that "the limitations of using dichotomous output indicators remain to be investigated" and that, "Despite the impressive results that have been obtained by investigators using dichotomous outputs, we retain a strong preference for quantitative output indicators until more adequate Monte Carlo tests have been done" (Meehl & Yonce, 1996, p. 1114). The findings of Maraun et al. would appear to substantiate Meehl's concerns.

For the case of continuous (quantitative) indicators, Meehl and colleagues have presented a reasoned argument (herein called "Meehl's Hypothesis"), and extensive Monte Carlo work, in support of the claim that MAXCOV can be used to make correct decisions about hypotheses of existence of T-structures. However, as with the dichotomous case, formal proofs of (R1) and (R2) have not, to date, been provided. It is the purpose of the present work to further understanding of MAXCOV for the case of continuous indicators through an analytic investigation of Meehl's Hypothesis, and the development of results on the behavior of the conditional covariance functions of T-structures.

THE LOGIC OF MAXCOV

Meehl derived MAXCOV on the basis of a characterization of T-structures. From his many discussions of MAXCOV (e.g., Meehl, 1973, 1992), it may be deduced that this characterization involves three elements, herein called M1, M2, and M3.

M1: Taxon and Complement Class

There exist two (latent) classes of individuals, one class called the taxon (T) and the other, the complement class (T'). This situation may be represented by defining $\boldsymbol{\theta}$, a latent variate, to be a random variate with Bernoulli distribution, such that

$$0 < P(\boldsymbol{\theta} = T) = \pi_T < 1, \text{ and } P(\boldsymbol{\theta} = T') = (1 - \pi_T), \tag{1}$$

a property that will, hereafter, be referred to as M1.

M2: Indicators

Define an "indicator" of T to be a continuous random variate, X_i, with the property that, after appropriate recoding,

$$P(X_i > x_i|\boldsymbol{\theta} = T) > P(X_i > x_i|\boldsymbol{\theta} = T'), \text{ for all values } x. \tag{2}$$

The MAXCOV procedure requires $p \geq 3$ such indicators, and these will be stored in the random vector $\underline{\mathbf{X}}$. Property (2) is known as positive regression dependence or stochastic ordering (Lehmann, 1966; Tukey, 1958) and, within the domain of latent variable modeling (e.g., Holland & Rosenbaum, 1986), "latent monotonicity."[1]

M3: Conditional Independence

One interpretation given by Meehl to the idea of a latent taxon is that it is a cause of the responding of individuals to the indicators. He paraphrases this notion in the usual way: The association that exists among the indicators is completely "explained" by the existence of the latent taxon and complement classes. For the case of continuous indicators, M3 states that

$$f_{\mathbf{X}|\theta=t} = \prod_{i=1}^{p} f_{X_i|\theta=t}, \tag{3}$$

that is, the joint density of the indicators conditional on $\boldsymbol{\theta} = t$, $t = \{T', T\}$, is a product of the individual conditional densities. It follows from Equation 3 that the two p by p conditional covariance matrices, $C(\underline{\mathbf{X}}|\boldsymbol{\theta} = T) = \Phi_T$ and $C(\underline{\mathbf{X}}|\boldsymbol{\theta} = T') = \Phi_{T'}$, are diagonal matrices. This diagonality condition, Meehl acknowledges, "is an idealization that will rarely be satisfied in MAXCOV-HITMAX applications" (Waller & Meehl, 1998, p. 17), but whose failure to obtain, he claims, "only rarely vitiates MAXCOV-HITMAX parameter estimates" (p. 17). In keeping with Meehl's ter-

[1]While latent monotonicity is the standard in latent variable modeling, it appears that Meehl defines "indicator" according to the weaker condition that $E(X_i|\boldsymbol{\theta} = T) > E(X_i|\boldsymbol{\theta} = T')$, the two senses equivalent only for the case of dichotomous variates. We begin with the stronger condition (2), but later discuss a possible justification for Meehl's choice.

minology, latent structures of the form [M1∩M2 ∩M3] will be said to comprise the class of taxonic structures (T-structures).

Now, in the psychometric literature, T-structures are called latent profile structures (for the case of dichotomous indicators, latent class structures). There exists a comprehensive theory on the estimation of the parameters of such structures (see, e.g., Bartholomew & Knott, 1999). Moreover, latent profile structures are members of the class of UMLV structures, and a great deal is known about the manifest properties that UMLV structures imply (see, e.g., Holland & Rosenbaum, 1986). Meehl has claimed that the conditional covariance functions of T-structures are single-peaked, and, hence, that this property can be used in the detection of T-structures. It is this possibility that makes the MAXCOV procedure of interest.

Rather than provide a direct proof of (R1), Meehl has provided, in a series of articles (e.g., Meehl, 1965, 1973, 1992; Meehl & Golden, 1982; Waller & Meehl, 1998), an argument as to why the conditional covariance functions of T-structures should be single-peaked and a number of supporting Monte Carlo studies. The statistical backdrop to his argument is as follows:

1. Partition $\underline{\mathbf{X}}$ as $[\mathbf{X}_{1(i)}, \mathbf{X}_{2(j)}, \underline{\mathbf{X}}^*]$, in which $\mathbf{X}_{1(i)}$ and $\mathbf{X}_{2(j)}$ are any two choices, $i \neq j$, from $\{\mathbf{X}_1, ..., \mathbf{X}_p\}$, and $\underline{\mathbf{X}}^*$ contains the $(p-2)$ remaining indicators.[2]
2. Define the random variate $\mathbf{X}_+ = \underline{1}'\underline{\mathbf{X}}^*$, that is, define it to be the sum of the $(p-2)$ indicators contained in $\underline{\mathbf{X}}^*$. Since $\mathbf{X}_+$ is the sum of $(p-2)$ indicators, it too is an indicator of T as defined in Equation 2.[3]
3. Indicator $\mathbf{X}_+$ is, in Meehl's terminology, the "input indicator," and indicators $\mathbf{X}_{1(i)}$ and $\mathbf{X}_{2(j)}$, the "output indicators" (Meehl & Yonce, 1996, p. 1097).
4. Define: $\pi_{\mathrm{Th}} = P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$; $\Phi_{\mathrm{Th}} = C([\mathbf{X}_{1(i)}, \mathbf{X}_{2(j)}]|\mathbf{X}_+ = h \cap \boldsymbol{\theta} = T)$ and $\Phi_{\mathrm{T'h}} = C([\mathbf{X}_{1(i)}, \mathbf{X}_{2(j)}]|\mathbf{X}_+ = h \cap \boldsymbol{\theta} = T')$, each a 2 by 2 conditional covariance matrix of $\mathbf{X}_{1(i)}$ and $\mathbf{X}_{2(j)}$; $\underline{\mu}_{\mathrm{Th}}$ a 2 by 1 vector with elements $E(\mathbf{X}_{1(i)}|\mathbf{X}_+ = h \cap \boldsymbol{\theta} = T)$ and $E(\mathbf{X}_{2(j)}|\mathbf{X}_+ = h \cap \boldsymbol{\theta} = T)$; and $\underline{\mu}_{T'h}$ a 2 by 1 vector with elements $E(\mathbf{X}_{1(i)}|\mathbf{X}_+ = h \cap \boldsymbol{\theta} = T')$ and $E(\mathbf{X}_{2(j)}|\mathbf{X}_+ = h \cap \boldsymbol{\theta} = T')$. The 2 by 2 covariance matrix of $\mathbf{X}_{1(i)}$ and $\mathbf{X}_{2(j)}$ conditional on $\mathbf{X}_+ = h$ can be expressed as:[4]

$$C\left\{\left[\mathbf{X}_{1(\mathrm{i})}, \mathbf{X}_{2(\mathrm{j})}\right] \middle| \mathbf{X}_+ = h\right\} =$$

$$\pi_{Th}\Phi_{Th} + \left(1-\pi_{Th}\right)\Phi_{T'h} + \pi_{Th}\left(1-\pi_{Th}\right)\left(\underline{\mu}_{Th} - \underline{\mu}_{T'h}\right)\left(\underline{\mu}_{Th} - \underline{\mu}_{T'h}\right)'. \tag{4}$$

[2]Note that $\underline{\mathbf{X}}^*$ may contain one or more variates. Following Gangestad and Snyder (1985), it has become popular when using dichotomous indicators to have $\underline{\mathbf{X}}^*$ contain more than one indicator. In Meehl's treatment of the continuous case, $\underline{\mathbf{X}}^*$ contains one variate. While this issue has no bearing on the generality of the key results, herein, presented, it will become clear in the course of the current investigation that it is useful to allow $\underline{\mathbf{X}}^*$ to contain multiple indicators.

[3]This fact is proven in many sources from the Psychometriks literature, including Lemma 13 of van der Linden (1998).

[4]A proof is contained in the Appendix.

Meehl's argument can then be paraphrased as follows[5] (see, e.g., Meehl & Yonce, 1996, pp. 1096–1097).

Given a T-structure, that is, a latent structure of the form [M1$\cap$M2$\cap$M3]:

1. The vector of mean differences, $(\underline{\mu}_{Th} - \underline{\mu}_{T'h})$, should be constant over the range of $\mathbf{X}_+$.

2. The indicators $\mathbf{X}_1$ and $\mathbf{X}_2$ should be statistically independent when conditioned on both $\mathbf{X}_+ = h$ and $\boldsymbol{\theta} = t$, and, hence, Φ_{Th} and $\Phi_{T'h}$ should be diagonal.

3. If 1. and 2. are correct, then the off-diagonal element of Equation 4, that is, $C\left(\mathbf{X}_1, \mathbf{X}_2 \,|\mathbf{X}_+ = h\right) = \pi_{Th}\left(1 - \pi_{Th}\right)\left(\mu_{1Th} - \mu_{1T'h}\right)\left(\mu_{2Th} - \mu_{2T'h}\right)$, will vary with h only through $\pi_{\mathrm{Th}}(1 - \pi_{\mathrm{Th}})$.

4. π_{Th} should be nondecreasing in h and should cross .5. Because $0 < \pi_{\mathrm{Th}} < 1$, $\pi_{\mathrm{Th}}(1 - \pi_{\mathrm{Th}})$ should then be a single-peaked function of h, with a maximum at $\pi_{\mathrm{Th}} = .5$.

Conclusion: T-structures yield single-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$.

The hypothesis that T-structures yield the properties described in 1.–4., that is,

5. (a) $(\underline{\mu}_{Th} - \underline{\mu}_{T'h})$ is constant over the range of $\mathbf{X}_+$, (b) Φ_{Th} and $\Phi_{T'h}$ are diagonal, (c) $\pi_{\mathrm{Th}} = P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ is a nondecreasing function of h, and (d) π_{Th} crosses .5, will, herein, be called "Meehl's Hypothesis." The hypothesis [T-structure]$\rightarrow$[$C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ single peaked] will be called (R1). Clearly, Meehl's Hypothesis is not necessarily equivalent to (R1), because, while 5. (a)–(d) are sufficient for [$C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ single peaked], it is not clear that they are necessary. Meehl's Hypothesis has the single-peakedness of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ being brought about in a very particular way, that is, according to 5. (a)–(d), but even if Meehl's Hypothesis is incorrect, (R1) might, nevertheless, be correct. That is, T-structures might yet necessarily yield single-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ but for different reasons than those described by 1.–4. It must be emphasized, however, that Meehl has offered no *other* rationale as to why T-structures should produce a single-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$.

Meehl also claims that a C-structure will produce a $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ that is flat over the range of $\mathbf{X}_+$. In his words: "If the latent structure is not taxonic, the curve will be flat" (Meehl, 1992, p. 134); "In MAXCOV-HITMAX the factorial situation does not give a dish ... but a flat graph" (Meehl, 1995, p. 272). Hence, according to Meehl, a single-peaked conditional covariance function distinguishes C- from T-structures. In particular, if Meehl is correct, evidence, in a given context, that $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is not single-peaked is evidence that the data did not arise from a T-structure. On the other hand, evidence that $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is single-peaked is evidence that the data did not arise from a C-structure.

[5]In the mathematics that follow, the more precise notations $\mathbf{X}_{1(\mathrm{i})}$ and $\mathbf{X}_{2(\mathrm{j})}$ are abandoned in favour of the more compact $\mathbf{X}_1$ and $\mathbf{X}_{\mathrm{j}}$.

In practice, a set of p indicators will yield $\frac{1}{2}p(p-1)$ unique input indicator/output indicator partitions, and, hence, the same number of empirical conditional covariance curves. A MAXCOV analysis involves a consideration of these curves, and, if the decision is made that they are in keeping with the hypothesis that the data arose from a T-structure, the estimation of a variety of important parameters including the base rate π_T, and Bayesian estimates of the probability of taxon membership. Meehl takes the agreement in the parameter estimates yielded by each partition to be further support ("consistency tests") for the taxonic conjecture.

For neither Meehl's Hypothesis, nor (R1), nor (R2), has Meehl provided formal proofs; he relies instead on evidence culled from extensive Monte Carlo work. However, the Monte Carlo study of Meehl and Golden (1982) did not give direct consideration to conditional covariance plots, but, rather, MAXCOV's ability to recover known values of the parameters of known T-structures. The study of Meehl and Yonce (1996) provided Monte Carlo generated conditional covariance plots under various conditions, the study as a whole appearing to support the claim that T-structures produce single-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$. The aim of the current effort is to gain further insight into the operation of the MAXCOV procedure for continuous indicators by developing results on the behaviour of the conditional covariance functions produced by T-structures, and, in particular, the truth of Meehl's Hypothesis.

MEEHL'S HYPOTHESIS

Meehl's Hypothesis is correct if, for any partition $[\mathbf{X}_1, \mathbf{X}_2, \underline{\mathbf{X}}^*]$ of $\underline{\mathbf{X}}$, the implication $[\text{M1}\cap\text{M2}\cap\text{M3}]\rightarrow[(5a)\cap(5b)\cap(5c)\cap(5d)]$ is true. Each component implication will be addressed in turn.

Theorem 1

If $[\text{M1}\cap\text{M3}]$, then, for any partition $[\mathbf{X}_1, \mathbf{X}_2, \underline{\mathbf{X}}^*]$ of $\underline{\mathbf{X}}$, $(\underline{\mu}_{Th} - \underline{\mu}_{T'h})$ is constant over h, that is, $[\text{M1}\cap\text{M3}]\rightarrow(5a)$ is true.

Proof. The v^{th} element of the 2 by 1 vector $(\underline{\mu}_{Th} - \underline{\mu}_{T'h})$ is equal to $E(\mathbf{X}_v|\mathbf{X}_+ = h \cap \boldsymbol{\theta} = T) - E(\mathbf{X}_v|\mathbf{X}_+ = h \cap \boldsymbol{\theta} = T')$, which, in turn, is equal to

$$\frac{\int_{-\infty}^{+\infty} \mathbf{X}_v f_{(\mathrm{X}_v\mathrm{X}_+=h)|\theta=T}\, \pi_T\, d(X_v)}{f_{(\mathrm{X}_+,\theta)}(h,T)} - \frac{\int_{-\infty}^{+\infty} \mathbf{X}_v f_{(\mathrm{X}_v\mathrm{X}_+=h)|\theta=T'}\,(1-\pi_T)\,d(X_v)}{f_{(\mathrm{X}_+,\theta)}(h,T')}. \quad (6)$$

From M3, the left member of Equation 6 is equal to

$$\frac{\int_{-\infty}^{+\infty} \mathbf{X}_v f_{X_v|\theta=T} f_{X_+|\theta=T}(h)\pi_T d(X_v)}{f_{(X_+,\theta)}(h,T)}, \tag{7}$$

and the right member to

$$\frac{\int_{-\infty}^{+\infty} \mathbf{X}_v f_{X_v|\theta=T'} f_{X_+|\theta=T'}(h)(1-\pi_T)d(X_v)}{f_{(X_+,\theta)}(h,T')}. \tag{8}$$

Since $f_{(X_+|\theta=t)}(h)\pi_t = f_{(X_+,\theta)}(h,t)$, Equation 6 is then equivalent to $E(\mathbf{X}_v|\boldsymbol{\theta} = T)$ $-E(\mathbf{X}_v|\boldsymbol{\theta} = T')$. Hence, $(\underline{\mu}_{Th} - \underline{\mu}_{T'h})$ is constant over h.

■

Theorem 2

If [M1∩M3], then, for any partition $[\mathbf{X}_1, \mathbf{X}_2, \underline{\mathbf{X}}^*]$ of $\underline{\mathbf{X}}$, the covariance matrices Φ_{Th} and $\underline{\Phi}_{T'h}$, are diagonal for all h, that is, [M1∩M3]→(5b).

Proof. The off-diagonal element of Φ_{th} is $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h, \boldsymbol{\theta} = t)$ which is equal to

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{X}_1\mathbf{X}_2 f_{(X_1,X_2)|X_+=h,\theta=t} d(X_1)d(X_2) - E\big(\mathbf{X}_1\big|\mathbf{X}_+=h,\theta=t\big)E\big(\mathbf{X}_2\big|\mathbf{X}_+=h,\theta=t\big). \tag{9}$$

It was already established that the right member of Equation 9 is equal to $E(\mathbf{X}_1|\boldsymbol{\theta} = t)E(\mathbf{X}_2|\boldsymbol{\theta} = t)$. Now, the left member is equal to

$$\frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{X}_1\mathbf{X}_2 f_{(X_1,X_2,X_+)|\theta=t}(h)\pi_t d(X_1)d(X_2)}{f_{(X_+,\theta)}(h,t)}, \tag{10}$$

which, from M3, is equal to

$$\frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{X}_1\mathbf{X}_2 f_{X_1|\theta=t} f_{X_2|\theta=t} f_{X_+|\theta=t}(h)\pi_t d(X_1)d(X_2)}{f_{(X_+,\theta)}(h,t)}, \tag{11}$$

which, since $f_{(X_+|\theta=t)}(h)\pi_t = f_{(X_+,\theta)}(h,t)$, is equal to $E(\mathbf{X}_1|\boldsymbol{\theta} = t)E(\mathbf{X}_2|\boldsymbol{\theta} = t)$. Hence, for all h, $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h \cap \boldsymbol{\theta} = t) = 0$, and the theorem is proven. ■

It follows immediately from Equation 4, and Theorems 1 and 2, that T-structures yield single-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ if and only if they yield single-peaked $\pi_{Th}(1 - \pi_{Th})$, the latter issue resting on the behaviour of $\pi_{Th} = P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$. Meehl's Hypothesis claims that T-structures yield $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ that are monotone nondecreasing and cross .5, whereby, if true, $\pi_{Th}(1 - \pi_{Th})$, and, hence, $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$, would indeed be single-peaked. On the other hand, it is now clear that (R1) is true so long as T-structures necessarily yield single-peaked $\pi_{Th}(1 - \pi_{Th})$. Potentially, $\pi_{Th}(1 - \pi_{Th})$ could be single-peaked for at least the following reasons: (a) $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ is monotone increasing (decreasing) and crosses .5, and (b) $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ is quadratic but does not cross .5. As it stands, it is not clear what restrictions T-structures place on the behaviour of $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$. The aim of the remainder of the article is to investigate this issue and, thereby, deduce useful results with respect the behaviour of the conditional covariance functions produced by T-structures.

What remains with respect to the analysis of Meehl's Hypothesis are issues (5c) and (5d), that is, whether $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ is an increasing function of h, and, if so, whether it crosses .5. To address (5c), the following definition and lemma are needed.

Definition (Monotone Likelihood Ratio Dependence, MLRD; Lehmann, 1966)

Two random variates **X** and **Y** (or their distribution) are (positive) monotone likelihood ratio dependent only if $f_{x,y}(x,y)f_{x,y}(x',y') \geq f_{x,y}(x,y')f_{x,y}(x',y)$ for all $x' > x, y' > y$ or, equivalently, $f_{x|y=y}(x)f_{x|y=y'}(x') \geq f_{x|y=y'}(x)f_{x|y=y}(x')$, in which $f_{x,y}(x, y)$ is the joint density of **X** and **Y**, and $f_{x|y=y}(x)$ is the conditional density of **X** given **Y** = y.

Lemma

$P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ is a nondecreasing function of h if and only if $\boldsymbol{\theta}$ and $\mathbf{X}_+$ are (positive) MLRD.

Proof. (Necessity) If $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ is a nondecreasing function of h, then, for any $\epsilon > 0$ and h, $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h + \epsilon) \geq P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$. Because

$$P\left(\theta = T\middle|\mathbf{X}_+ = h\right) = \frac{f_{X_+|\theta=T}(h)\pi_T}{f_{X_+}(h)}, \tag{12}$$

what must be shown is that

$$\frac{f_{X_+|\theta=T}(h+\epsilon)\pi_T}{f_{X_+}(h+\epsilon)} \geq \frac{f_{X_+|\theta=T}(h)\pi_T}{f_{X_+}(h)}. \tag{13}$$

However,

$$f_{X_+}(h) = \pi_T * f_{X_+|\theta=T}(h) + (1-\pi_T)* f_{X_+|\theta=T'}(h) \tag{14}$$

so Equation 13 is equivalent to

$$f_{X_+|\theta=T}(h+\epsilon) f_{X_+|\theta=T'}(h) \geq f_{X_+|\theta=T}(h) f_{X_+|\theta=T'}(h+\epsilon), \tag{15}$$

in which $f_{X_+|\theta=t}(s)$ is the value, when $\mathbf{X}_+ = s$, of the conditional density of $\mathbf{X}_+$ given $\boldsymbol{\theta} = t$. This is the requirement that $\mathbf{X}_+$ and $\boldsymbol{\theta}$ be MLRD.

(Sufficiency) The argument proceeds in reverse.

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MLRD is a strong form of dependence which, as is clear from its definition, is induced by the joint distribution of two variates. It certainly is not a given that MLRD will characterize the joint distribution of an arbitrary pair of random variates (see, e.g., Karlin & Rinott, 1980). Lehmann (1966) provided a number of examples of distributions that are MLRD. Within the context of item response theory, the MLRD property has been discussed by Holland and Rosenbaum (1986) (under the heading of TP_2 distribution); Grayson (1988); Huynh (1994); and Hemker, Sijtsma, Molenaar, and Junker (1997). The following theorem establishes that [M1∩M2∩M3] does not imply that $\boldsymbol{\theta}$ and $\mathbf{X}_+$ are MLRD.

Theorem 3

[M1∩M2∩M3] does not imply that $\mathbf{X}_+$ and $\boldsymbol{\theta}$ are MLRD and, hence, does not imply that $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ is a nondecreasing function of h, that is, [M1∩M2∩M3] does not imply (5c).

Proof. Lemma 2 of Holland and Rosenbaum (1986) establishes that [M1∩M2∩M3] induces not MLRD, but a weaker form of dependency in the joint distribution of $[\boldsymbol{\theta}, \underline{\mathbf{X}}^*]$, namely that, for any increasing function of $\underline{\mathbf{X}}^*$, say, $g(\underline{\mathbf{X}}^*)$, $E[g(\underline{\mathbf{X}}^*)|\boldsymbol{\theta} = t)$ is a nondecreasing function of t.

■

Theorem 3 establishes that Meehl's Hypothesis is false, because it establishes that [M1∩M2∩M3] does not imply that $\mathbf{X}_+$ and $\boldsymbol{\theta}$ are MLRD, and, hence, that T-structures do not necessarily produce $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ that are nondecreasing functions of h. In his descriptions of MAXCOV, Meehl has frequently spoken of the assumption that the densities $f_{X_+|\theta=T}(s)$ and $f_{X_+|\theta=T'}(s)$ are either each unimodal or cross only once (e.g., Meehl & Yonce, 1996, p. 1097). However, the unimodality of each of $f_{X_+|\theta=T}(s)$ and $f_{X_+|\theta=T'}(s)$ does not imply that they will cross only once, nor is the converse implication true. More essentially, neither the unimodality of each of the densities $f_{X_+|\theta=T}(s)$ and $f_{X_+|\theta=T'}(s)$ nor their crossing only once, implies that $\mathbf{X}_+$ and $\boldsymbol{\theta}$ are MLRD, and, hence, that $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ will be a nondecreasing function of h. It will later be shown that even if $f_{X_+|\theta=T}(s)$ and $f_{X_+|\theta=T'}(s)$ are each normal densities, it does not follow that $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ is a nondecreasing function of h. As will be seen, the case of conditional normality is particularly simple, because whether or not $\mathbf{X}_+$ and $\boldsymbol{\theta}$ are MLRD is determined by moments of order two and lower. For more complicated conditional densities of the kind Meehl has suggested might arise in applications of MAXCOV, the issue as to whether or not $\mathbf{X}_+$ and $\boldsymbol{\theta}$ are MLRD will be all the more complicated, resting as it will on higher order moments.

Hemker et al. (1997) proved that MLRD of $\mathbf{X}_+$ and continuous $\boldsymbol{\theta}$ is not, in general, implied by unidimensional, monotone item response structures for polytomous items. Interestingly, MLRD *is* implied by [M1∩M2∩M3] in the case of dichotomous indicators (Maraun et al., 2003), in agreement with Holland and Rosenbaum's (1986) finding that dichotomous indicators of UMLV structures manifest stronger dependencies than do their continuous counterparts. Theorem 3 does not, of course, imply that (R1) is false, because $\pi_{Th}(1 - \pi_{Th})$ may yet be necessarily single-peaked. The status of (R1) remains unclear.

MEEHL'S MONTE CARLO SUPPORT

The Monte Carlo study of Meehl and Yonce (1996), which involved taxonic data sets constructed in accord with Meehl's account of taxonicity, seems to provide support for Meehl's Hypothesis. Yet, Theorem 3 shows Meehl's Hypothesis to be false. Possible explanations for this apparent discrepancy include the following: (a) (R1) is true, but does not come about as suggested by Meehl's Hypothesis. That is, $\pi_{Th}(1 - \pi_{Th})$ is necessarily single-peaked, but not because $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ necessarily is nondecreasing and crosses .5, and (b) there exists a set of conditions, say, $\{t_1, t_2, ...\}$, that are features of the Monte Carlo simulations of Meehl and Yonce, and for which the implication $[\{M1 \cap M2 \cap M3\} \cap t_1 \cap t_2 \cap ...] \rightarrow [(5a) \cap (5b) \cap (5c) \cap (5d)]$ is true. While not discounting the possibility of (a), it is shown in this section that at least explanation (b) is correct. That is, it is shown that features of the data generation protocol inherent to the study of Meehl and Yonce make Meehl's Hypothesis appear to be true.

To begin, consider the means by which Meehl and Yonce (1996) generated their data, N (subject) by 4 (indicator) matrices sampled from populations in which the indicators conform to either a C-structure (in this case, a unidimensional linear factor structure) or T-structure, under various combinations of parameter values. The recipe for the creation of these data sets is found on pages 1066 to 1068 of Meehl and Yonce and may be described as follows:

For each data set:

1. Generate an $N \times 5$ matrix, $\mathbf{Z}$, each row a realization of a multivariate normal random vector, $\underline{\mathbf{z}} \sim N(\underline{0}, I)$.[6]
2. Partition $\mathbf{Z}$ as $[\underline{\mathbf{y}}|\mathbf{E}]$, in which $\underline{\mathbf{y}}$ is an N by 1 vector, and $\mathbf{E}$, an N by 4 matrix.
3. Construct the N by 4 matrix, $\mathbf{M}$, as follows:
   Taxonic data sets: $\mathbf{M} = \underline{\mathbf{y}} * \underline{\mathbf{1}}_4' * .001$
   Nontaxonic data sets: $\mathbf{M} = \underline{\mathbf{y}} * \underline{\lambda}'$
   in which $\underline{\mathbf{1}}_4$ is a four element vector containing unities, and $\underline{\lambda}'$ is chosen so that the variates constructed under the taxonic and nontaxonic scenarios will have identical covariance matrices.
4. For the taxonic data sets, assign the first N_{T} individuals to the taxon, and the final $\left(N - N_T\right) = N_{T'}$ individuals to the complement class. Define γ to be the "separation" of the two classes, that is, their mean difference, and redefine $\mathbf{M}$ to be:

$$\mathbf{M} = \underline{\mathbf{y}} * \underline{\mathbf{1}}_4' * .001 + \gamma \begin{bmatrix} 1_{N_{\mathrm{T}}} \\ 0_{N_{\mathrm{T}'}} \end{bmatrix} \underline{\mathbf{1}}_4'.$$

5. Finally, for both taxonic and nontaxonic data sets, create the final $N \times 4$ matrix $\mathbf{X}$, whose columns each contain an "indicator," as

$$\mathbf{X} = \mathbf{M} + \mathbf{E}.$$

Matrix $\mathbf{X}$ then contains items whose latent structure is either a T- or C-structure, but which are perturbed by the error matrix $\mathbf{E}$. To study the impacts of the parameters N_{T} and γ on MAXCOV's ability to detect T-structures, Meehl and Yonce (1996) vary these parameters over data sets.

Twenty-five data sets under "…each of various taxonic and nontaxonic configurations" (Meehl & Yonce, 1996, p. 1098), were generated, and, for each data set, variate $\mathbf{X}_+$, the input indicator, was set, in turn, to each of the four items in $\mathbf{X}$. Successive intervals were demarcated along $\mathbf{X}_+$, and then, within each interval, the three pairwise covariances involving the remaining three variates were calculated. Importantly, the data generation protocol employed by Meehl and Yonce ensured that, for the taxonic samples, the conditional variances of $\mathbf{X}_+$ given T, and given T',

[6]As far as we can tell, Meehl and Yonce do not actually provide information regarding the moments of this random vector. However, $N(\underline{0}, I)$ seems to square with the rest of their description.

were equal, and that the conditional distributions of $\mathbf{X}_+$ given T, and given T', were each normally distributed.[7] Theorems 4 and 5 establish that [M1∩M2∩M3] in conjunction with conditional normality and equality of conditional variances does imply [(5a)∩(5b)∩(5c)∩(5d)].

Theorem 4

Given [M1∩M2∩M3] and that $f_{X_+|\theta=t}(h),\ t=\{T',T\}$, are each normal densities, a necessary and sufficient condition that $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h)$ is nondecreasing over h, that is, that (5c) is true, is that $\sigma^2_{X_+|T} = \sigma^2_{X_+|T'}$.

Proof. Let $f_{X_+|\theta=t}(h),\ t=\{T',T\}$, each be a normal density, that is,

$$\frac{1}{\left(2\pi\sigma^2_{X_+|t}\right)^{\frac{1}{2}}}\exp\left[-\frac{\left(h-\mu_{X_+|t}\right)^2}{2\sigma^2_{X_+|t}}\right], \tag{16}$$

in which $\mu_{X_+|t} = E\left(\mathbf{X}_+|\theta=t\right)$ and $\sigma^2_{X_+|t} = E\left[\left(\mathbf{X}_+ - \mu_{X_+|t}\right)^2|\theta=t\right]$. From lemma 2 of Holland and Rosenbaum (1986), [M1∩M2∩M3] implies that, for all j, $E(\mathbf{X}_j|\boldsymbol{\theta} = T) > E(\mathbf{X}_j|\boldsymbol{\theta} = T')$. It follows, then, that $\mu_{X_+|T} > \mu_{X_+|T'}$, and, hence, that $\mu_{X_+|T} = \mu_{X_+|T'} + \gamma,\ \gamma > 0$. Recall that Inequality 15 must be satisfied in order that $\boldsymbol{\theta}$ and $\mathbf{X}_+$ be MLRD, and, hence, $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ be nondecreasing in h. Substituting Expression 16 into Inequality 15, and taking the natural logarithm of both sides of the inequality results in the condition that

$$\frac{\left(h+\epsilon-\mu_{X_+|T'}\right)^2}{\sigma^2_{X_+|T'}} + \frac{\left(h-\mu_{X_+|T'}-\gamma\right)^2}{\sigma^2_{X_+|T}} - \frac{\left(h+\epsilon-\mu_{X_+|T'}-\gamma\right)^2}{\sigma^2_{X_+|T}} - \frac{\left(h-\mu_{X_+|T'}\right)^2}{\sigma^2_{X_+|T'}}, \tag{17}$$

must be nonnegative for all h and $\epsilon > 0$. Expansion of Expression 17 yields

$$-\frac{2\epsilon\left(\sigma^2_{X_+|T'}-\sigma^2_{X_+|T}\right)h}{\sigma^2_{X_+|T'}\sigma^2_{X_+|T}} - \frac{\epsilon\left(\epsilon\sigma^2_{X_+|T'}-\epsilon\sigma^2_{X_+|T}-2\gamma\sigma^2_{T'}-2\mu_{X_+|T'}\sigma^2_{X_+|T'}+2\mu_{X_+|T'}\sigma^2_{X_+|T}\right)}{2\sigma^2_{X_+|T'}\sigma^2_{X_+|T}}. \tag{18}$$

[7]Four of the data sets of Meehl and Golden (1982) involved unequal conditional variances, but their study did not include an examination of the behavior of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$.

Replacing both $\sigma^2_{X_+|T}$ and $\sigma^2_{X_+|T'}$ with σ^2 reduces Expression 18 to

$$\frac{2\gamma\epsilon}{\sigma^2},$$

which is positive.

Because the second term of Expression 18 is a constant with respect to h, and h ranges over the entire real line, if Expression 18 is nonnegative for all h, then the first term of Expression 18, a linear function of h, must be equal to zero. This will be the case only if $\sigma^2_{X_+|T} = \sigma^2_{X_+|T'}$.

■

When, in addition to [M1∩M2∩M3], $f_{X_+|\theta=t}$ is a normal distribution for each of T' and T, and $\sigma^2_{X_+|T} = \sigma^2_{X_+|T'}$, it follows that the distribution of [$\mathbf{X}_+$, $\boldsymbol{\theta}$] is MLRD, and $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ is a nondecreasing function of h. Interestingly, given the normality of $f_{X_+|\theta=t}$, $t = \{T', T\}$, the brand of monotonicity described by (M2) can be replaced by the weaker brand it implies, and which is implicitly adopted by Meehl and Yonce (1996), namely that, for all j, $E(\mathbf{X}_j|\boldsymbol{\theta} = T) > E(\mathbf{X}_j|\boldsymbol{\theta} = T')$. For then, $\mu_{X_+|T} > \mu_{X_+|T'}$.

Theorem 5

Given [M1∩M2∩M3] and that $f_{X_+|\theta=t}$, $t = \{T', T\}$, are each normal densities, a necessary and sufficient condition that $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ crosses .5, that is, that (5d) is true, is that $\sigma^2_{X_+|T} = \sigma^2_{X_+|T'}$.

Proof. (Sufficiency) The requirement that $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ crosses .5 is equivalent to the requirement that

$$\frac{f_{X_+|\theta=T}(h)\pi_T}{f_{X_+}(h)}$$

crosses .5. Once again employing Equation 14, it may be shown that this requirement is equivalent to the requirement that

$$\frac{f_{X_+|\theta=T'}(h)}{f_{X_+|\theta=T}(h)},$$

crosses

$$\frac{\pi_T}{(1-\pi_T)}.$$

Taking the natural logarithm of both quantities, after using Expression 16, results in the condition that

$$c+\frac{h^2\left(\sigma^2_{X_+|T'}-\sigma^2_{X_+|T}\right)+2h\left(\mu_{X_+|T'}\sigma^2_{X_+|T}-\mu_{X_+|T}\sigma^2_{X_+|T'}\right)+\sigma^2_{X_+|T'}\mu^2_{X_+|T}-\sigma^2_{X_+|T}\mu^2_{X_+|T'}}{2\sigma^2_{X_+|T}\sigma^2_{X_+|T'}}, \quad (19)$$

must cross

$$\ln\left[\frac{\pi_T}{(1-\pi_T)}\right], \quad (20)$$

in which

$$c=\ln\sqrt{\frac{\sigma^2_{X_+|T}}{\sigma^2_{X_+|T'}}}.$$

Replacing both $\sigma^2_{X_+|T}$ and $\sigma^2_{X_+|T'}$ with σ^2 in Expression 19 results in

$$\frac{2h\left(\mu_{X_+|T'}-\mu_{X_+|T}\right)+\left(\mu^2_{X_+|T}-\mu^2_{X_+|T'}\right)}{2\sigma^2}, \quad (21)$$

a linear function of h, with negative slope

$$\frac{\left(\mu_{X_+|T'}-\mu_{X_+|T}\right)}{\sigma^2}.$$

Clearly then, since h ranges over the entire real line, Expression 19 will always cross Expression 20.

(Necessity) Assume that Expression 19 crosses Expression 20. Since the range of Expression 20 is the entire real line, this must mean that Expression 19 cannot have a minimum or maximum. This will be the case only if the quadratic term of Expression 19 disappears, that is, when $\sigma^2_{X_+|T}=\sigma^2_{X_+|T'}$.

■

Let the subclass of T-structures that are described as

$$\left\{[\mathrm{M1}\cap\mathrm{M2}\cap\mathrm{M3}]\cap\left[f_{X_+|\theta=t}\sim N\left(\mu_{X_+|\theta=t},\sigma^2_{X_+|\theta=t}\right),t=\{T',T\}\right]\cap\left(\sigma^2_{X_+|T}=\sigma^2_{X_+|T'}\right)\right\}$$

be called TNEV-structures. Members of the class of TNEV-structures vary with respect to the parameters $\left(\pi_T, \mu_{X_+|T}, \mu_{X_+|T'}, \sigma_c^2\right)$, in which σ_c^2 is the value of the common conditional variance. The findings to this point can be summarized as follows:

1. [M1∩M2∩M3] does not imply (5a)–(5d), that is, T-structures do not necessarily yield (5a)–(5d). Hence, if they necessarily yield single-peaked conditional covariance functions, they do not do so in the manner described by Meehl's Hypothesis;
2. $\left\{[\mathrm{M1} \cap \mathrm{M2} \cap \mathrm{M3}] \cap \left[f_{\mathrm{X}_+|\theta=t} \sim N\left(\mu_{\mathrm{X}_+|\theta=t}, \sigma^2_{\mathrm{X}_+|\theta=t}\right), t = \{T', T\}\right] \cap \left(\sigma^2_{\mathrm{X}_+|T} = \sigma^2_{\mathrm{X}_+|T'}\right)\right\} \rightarrow [(5a) \cap (5b) \cap (5c) \cap (5d)]$, that is, TNEV-structures necessarily yield single-peaked conditional covariance functions.

With respect to these findings, several observations can be made:

1. Because single peakedness of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is a necessary condition for TNEV-structures, evidence that $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is *not* single peaked in a given empirical context can be taken as evidence that the data did not arise from a TNEV-structure.

2. There is no a priori reason to believe that only TNEV-structures, which comprise a tiny fraction of all T-structures, will arise in applied settings. In the first place, conditional normality and homogeneity of variance are, according to Meehl, latent properties, and, hence, the researcher will not *know* whether they are likely to obtain in practice (this lack of knowledge presumably the reason he is interested in employing decision-making machinery such as MAXCOV). In the second place, Meehl has made clear that MAXCOV was developed for domains of application in which it can be expected that "skewness and heterogeneity of variance are common" (Meehl & Yonce, 1996, p. 1097). As he states, these are "assumptions which are not likely to obtain (and have frequently been shown *not* to obtain) when the domain of investigation is a clinical taxon such as schizotypy" (Meehl, 1968, p. 47). One might take these claims as suggesting that TNEV-structures are unlikely to arise in the contexts in which MAXCOV is standardly employed.

3. With reference to the Monte Carlo study of Meehl and Golden (1982), Meehl and Yonce (1996) stated that "unequal variances … have little effect on trustworthiness of estimations" (p. 1097). But the issue as to whether MAXCOV yields correct *decisions* about the hypothesis that "the data have arisen from a T-structure" is distinct from that of the accuracy of the estimates of the parameters it yields, *given* knowledge that these parameters are, in fact, the parameters of a T-structure. Meehl and Yonce also argued that

> although our Monte Carlo data were generated by a Gaussian algorithm assigning equal variances SD_t^2, SD_c^2 to taxon and complement classes, none of the core derivations underlying MAXCOV are thus restrictive. The conjectured structure … is

> highly general, that of two overlapping unimodal frequency distributions. The mathematics speaks for itself, and it was developed by Meehl with psychopathology in mind, where skewness and heterogeneity of variance are common. (Meehl & Yonce, 1996, p. 1097)

However, while the referred to structure, that is, the covariance structure of Equation 4, is indisputably “highly general,” it is, unfortunately, sufficient to make neither Meehl’s Hypothesis nor (R1) true. As was made clear in the lemma given prior to Equation 12, whether components (5c) and (5d) obtain is determined, not by properties of Equation 4, but, rather, by properties of the joint distribution of $[\underline{\mathbf{X}}^*, \boldsymbol{\theta}]$.

Once again, Meehl has never proven that the implication [T-structure] $\rightarrow$ [$C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is single peaked] is true, but, rather, has offered a rationale, herein called Meehl’s Hypothesis, as to why it should be true, and a set of Monte Carlo studies that he has taken as demonstrating that it is true. It turns out, however, that Meehl’s Hypothesis is false. It is not T-structures that yield (5a)–(5d), and, as a result, single-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$, but, rather, TNEV-structures. The presence of conditional normality and equality of conditional variances in Meehl’s Monte Carlo studies would undoubtedly have given the (false) appearance that Meehl’s Hypothesis was true.

Let the subclass of T-structures that are described as

$$\left\{[\mathrm{M1} \cap \mathrm{M2} \cap \mathrm{M3}] \cap \left[f_{\mathrm{X}_+|\theta=t} \sim N\left(\mu_{\mathrm{X}_+|\theta=t}, \sigma^2_{\mathrm{X}_+|\theta=t}\right), t = \{T', T\}\right]\right\}$$

be called TN-structures. Members of the class of TN-structures vary with respect the parameters $\left[\pi_T, \mu_{\mathrm{X}_+|T}, \mu_{\mathrm{X}_+|T'}, \sigma^2_{\mathrm{X}_+|T}\ \sigma^2_{\mathrm{X}_+|T'}\right]$. It has already been established that TNEV-structures, the subclass of TN-structures for which $\sigma^2_{\mathrm{X}_+|T} = \sigma^2_{\mathrm{X}_+|T'}$, yield single-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$. However, it remains unclear what can be said about the $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ necessarily yielded by members of the general class of T-structures. The remainder of the article is devoted to developing some insights into this question. The following Theorem will be needed.

Theorem 6

As $(p - 2) \rightarrow \infty$, T-structures converge to TN-structures.

Proof. From the central limit theorem, as $(p - 2)$, the number of indicators contained in $\underline{\mathbf{X}}^*$, becomes large, $f_{\mathrm{X}_+|\theta=t}$, $t = \{T', T\}$, will each converge to normal densities (Basawa & Rao, 1980; Holland, 1990).

■

It can be expected that reasonable approximations to conditional normality will obtain even for moderate $(p-2)$, say, in the range of 5. Thus, it can be expected that, for even moderate $(p-2)$, T-structures are, in essence, TN-structures. This fact can be used to gain insight into the properties of the conditional covariance functions of T-structures. The class of TN-structures is the union of the subclasses of TNEV-structures, and the unequal conditional variance structures, herein called TNUV-structures (i.e., TN = TNEV $\cup$ TNUV). Hence, for moderate $(p-2)$, the class of T-structures is comprised of TNEV- and TNUV-structures. It has already been shown that TNEV-structures yield single-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$. The aim is now to deduce the conditional covariance functions yielded by TNUV-structures.

CONDITIONAL COVARIANCE FUNCTIONS OF TNUV-STRUCTURES

It will be assumed in this section that $(p-2)$ is large enough to make every T-structure a TN-structure. The aim is then to deduce the properties of $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$, and, as a result, $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$, yielded by TNUV-structures. It will be convenient to begin by rearranging Equation 19 as

$$ah^2 + bh + d, \tag{22}$$

in which

$$a = \frac{\left(\sigma^2_{X_+|T'} - \sigma^2_{X_+|T}\right)}{2\sigma^2_{X_+|T}\sigma^2_{X_+|T'}},\ b = \frac{\left(\mu_{X_+|T'}\sigma^2_{X_+|T} - \mu_{X_+|T}\sigma^2_{X_+|T'}\right)}{\sigma^2_{X_+|T}\sigma^2_{X_+|T'}},$$

$$\text{and } d = \frac{\sigma^2_{X_+|T'}\mu^2_{X_+|T} - \sigma^2_{X_+|T}\mu^2_{X_+|T'}}{2\sigma^2_{X_+|T}\sigma^2_{X_+|T'}} + c.$$

$P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ can then be expressed as

$$\left[1 + \frac{(1-\pi_T)}{\pi_T}\exp\left(ah^2 + bh + d\right)\right]^{-1}, \tag{23}$$

and $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ as

$$\frac{\dfrac{(1-\pi_T)}{\pi_T}\exp\left(ah^2 + bh + d\right)}{\left[1 + \dfrac{(1-\pi_T)}{\pi_T}\exp\left(ah^2 + bh + d\right)\right]^2}\left(\mu_{1Th} - \mu_{1T'h}\right)\left(\mu_{2Th} - \mu_{2T'h}\right). \tag{24}$$

Thus, Equation 24 is the asymptotic conditional covariance function of the T-structure, i.e., the conditional covariance function of the T-structure, as $(p-2)$ becomes large. Setting to zero the derivative of Equation 23 with respect to h results in

$$2ah + b = 0. \tag{25}$$

Thus, $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ has a critical point if and only if $a \neq 0$, that is, the T-structure is a TNUV-structure, and this critical point, a maximum (minimum) if $\sigma^2_{X_+|T} < \sigma^2_{X_+|T'}$ $\left(\sigma^2_{X_+|T} > \sigma^2_{X_+|T'}\right)$, is located at

$$v = \frac{-b}{2a} = \frac{\mu_{X_+|T}\sigma^2_{X_+|T'} - \mu_{X_+|T'}\sigma^2_{X_+|T}}{\sigma^2_{X_+|T'} - \sigma^2_{X_+|T}}. \tag{26}$$

It follows, then, that TNUV-structures yield $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ that are quadratic functions of h. Moreover, if $\sigma^2_{X_+|T} < \sigma^2_{X_+|T'}$ $\left(\sigma^2_{X_+|T} > \sigma^2_{X_+|T'}\right)$, this quadratic becomes narrower as π_T decreases (increases) in (0, 1). Notice also that, if $\delta = \sigma^2_{X_+|T'} - \sigma^2_{X_+|T} > 0$, $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ will converge to zero as $h \to \infty$ and as $h \to -\infty$, and, hence, will be concave, while, if $\delta < 0$, it will converge to unity as $h \to \infty$ and as $h \to -\infty$, and, hence, be convex.

Thus, TNUV-structures yield $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ that have the following properties: (a) three critical points, say, $\{h_1, h_2, h_3\}$, if $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ crosses .5; (b) one critical point, $\{h_2\}$, if $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ does not cross .5; and, (c) converge to zero as $h \to \infty$ and as $h \to -\infty$. The critical points $\{h_1, h_2, h_3\}$, obtained by noting that, by (5a), $\left(\mu_{1Th} - \mu_{1T'h}\right)\left(\mu_{2Th} - \mu_{2T'h}\right)$ is a constant over the range of $\mathbf{X}_+$, and by setting the derivative of Equation 24 to zero, are equal to:

$$h_1 = \frac{\left(\mu_{X_+|T}\sigma^2_{X_+|T'} - \mu_{X_+|T'}\sigma^2_{X_+|T}\right) - \sigma^2_{X_+|T'}\sigma^2_{X_+|T}\sqrt{\dfrac{\gamma^2 - \delta\ln\left(\dfrac{\sigma^2_{X_+|T}}{\sigma^2_{X_+|T'}}\right) + 2\delta\ln\left[\dfrac{\pi_T}{(1-\pi_T)}\right]}{\sigma^2_{X_+|T'}\sigma^2_{X_+|T}}}}{\delta},$$

$$h_2 = v,$$

$$h_3 = \frac{\left(\mu_{X_+|T}\sigma^2_{X_+|T'} - \mu_{X_+|T'}\sigma^2_{X_+|T}\right) - \sigma^2_{X_+|T'}\sigma^2_{X_+|T}\sqrt{\dfrac{\gamma^2 - \delta\ln\left(\dfrac{\sigma^2_{X_+|T}}{\sigma^2_{X_+|T'}}\right) + 2\delta\ln\left[\dfrac{\pi_T}{(1-\pi_T)}\right]}{\sigma^2_{X_+|T'}\sigma^2_{X_+|T}}}}{\delta}. \tag{27}$$

It can be shown by substitution that, when $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ does cross .5, h_1 and h_3 are, indeed, the points at which it does so, and that at v, $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ is equal to

$$\left[1+\frac{1-\pi_T}{\pi_T}\exp\left(\frac{-b^2}{4a}+d\right)\right]^{-1}. \tag{28}$$

It follows then that, if $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ crosses .5, $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ has absolute maxima of .25 at h_1 and h_3, and a local minimum at v. If, on the other hand, $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ does not cross .5, $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ has an absolute maximum at v. It can then be concluded that TNUV-structures yield $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ that are:

2-peaked if $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ crosses .5;
1-peaked if $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ does not cross .5.

To understand the behaviour of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ yielded by TNUV-structures, the conditions under which these structures produce $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ that do, and do not, cross .5 must, then, be investigated. If $\sigma^2_{X_+|T} \neq \sigma^2_{X_+|T'}$, then $a \neq 0$, and Equation 22 will be a parabola with vertex located at $[v, \varphi]$, in which

$$\varphi=\frac{-\gamma^2+\delta\ln\left[\frac{\sigma^2_{\mathrm{X}_+|T}}{\left(\sigma^2_{\mathrm{X}_+|T}+\delta\right)}\right]}{2\delta}. \tag{29}$$

If a is positive, that is, $\sigma^2_{X_+|T'} > \sigma^2_{X_+|T}$ ($\delta > 0$), φ will then be a minimum of Equation 19, while if a is negative, that is, $\sigma^2_{X_+|T} > \sigma^2_{X_+|T'}$ ($\delta > 0$), it will be a maximum of Equation 19. It follows, then, that, because the range of Equation 20 is the entire real line, if a is positive, there will exist a subset of values of π_T, say, $\pi_T < ub$, for which Equation 20 will be less than this minimum, and, hence, $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h) < .5$ for all h. If a is negative, there will exist a subset of values of π_T, say $\pi_T > lb$, for which Equation 20 will be greater than this maximum, and, hence, $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h) > .5$ for all h. If $\delta > 0$ ($\delta < 0$), ub (lb) is the solution, in π_T, to the equation

$$\ln\left[\frac{\pi_T}{(1-\pi_T)}\right]=\varphi, \tag{30}$$

this solution being

$$\frac{\exp(\varphi)}{[1+\exp(\varphi)]}. \tag{31}$$

It follows from Equations 29 and 31 that *lb* lies in the interval (.5, 1] and *ub* in the interval [0, .5). It can then be deduced that:

1. TNUV-structures for which $\delta > 0$ yield $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ that are 2-peaked unless $\pi_T < ub < .5$, in which case they yield $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ that are single-peaked;
2. TNUV-structures for which $\delta < 0$ yield $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ that are 2-peaked unless $\pi_T > lb > .5$, in which case they yield $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ that are single-peaked;
3. TNUV-structures for which $\pi_T = .5$ yield $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ that cross .5, and, hence, $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ that are 2-peaked.

The partial derivatives of φ with respect to γ^2 and δ are, respectively,

$$-\frac{1}{2\delta}, \tag{32}$$

and

$$\frac{\gamma^2}{2\delta^2} - \frac{1}{2\sigma^2_{X_+|T}}. \tag{33}$$

From Equation 32, it may be concluded that if $\delta > 0$, φ and, hence, *ub*, are decreasing functions of γ^2, while if $\delta < 0$, φ and, hence, *lb*, are increasing functions of γ^2. Moreover, for fixed $\delta < 0$, as $\gamma^2 \to \infty$, $\varphi \to \infty$, and, hence, $lb \to 1$, while, for fixed $\delta > 0$, as $\gamma^2 \to \infty$, $\varphi \to -\infty$, and, hence, $ub \to 0$. Hence, for fixed δ, the better are the indicators, that is, the larger is the value of γ^2, the more extreme will π_T have to be before $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ fails to cross .5. Thus, TNUV-structures with high-quality indicators and/or a nonextreme value of π_T can be expected to yield $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ that are 2-peaked.

The behaviour of φ, and, hence, *lb*/*ub*, as a function of δ is more complicated. Let δ^* be equal to

$$\sqrt{\gamma^2\sigma^2_{X_+|T}}.$$

Because, from Equation 33, for fixed γ^2, if $|\delta| < \delta^*$ ($|\delta| > \delta^*$), φ is increasing (decreasing) in δ, the following may be deduced:

as $\delta \to -\sigma^2_{X_+|T}$, $\varphi_{\min} \to \infty$, and, hence, $lb \to 1$;

for $-\sigma^2_{X_+|T} < \delta < -\delta^*$, *lb* is decreasing in δ;

for $\delta = -\delta^*$, *lb* is at its minimum (with respect δ) of in which $\dfrac{\exp(\varphi_{\min})}{1+\exp(\varphi_{\min})}$

$$\varphi_{\min} = \frac{\gamma + \sigma^2_{X_+|T} \ln\left[1 - \dfrac{\gamma\sigma^2_{X_+|T}}{\left(\sigma^2_{X_+|T} - \delta^*\right)}\right]}{2\sigma^2_{X_+|T}};$$

for $-\delta^* < \delta < 0$, *lb* is increasing in δ;

for $\delta = 0$, *lb*/*ub* not defined;

for $0 < \delta < \delta^*$, *ub* is increasing in δ;

for $\delta = \delta^*$, *ub* is at its maximum (with respect of δ), $\dfrac{\exp(\varphi_{\max})}{1+\exp(\varphi_{\max})}$, in which

$$\varphi_{\max} = \frac{-\gamma + \sigma^2_{X_+|T} \ln\left[1 - \dfrac{\gamma\sigma^2_{X_+|T}}{\left(\sigma^2_{X_+|T} - \delta^*\right)}\right]}{2\sigma^2_{X_+|T}};$$

for $\delta^* < \delta$, *ub* is decreasing in δ;

as $\delta \to \infty$, $\varphi_{\max} \to -\infty$, and $ub \to 0$. (34)

Figure 1 depicts the behaviour of *lb*/*ub* as a function of δ for the case in which $\gamma = .81$ and $\sigma^2_{X_+|T} = 10$. Because, in this case, $\delta^* = 2.56$, the curve on the left of the

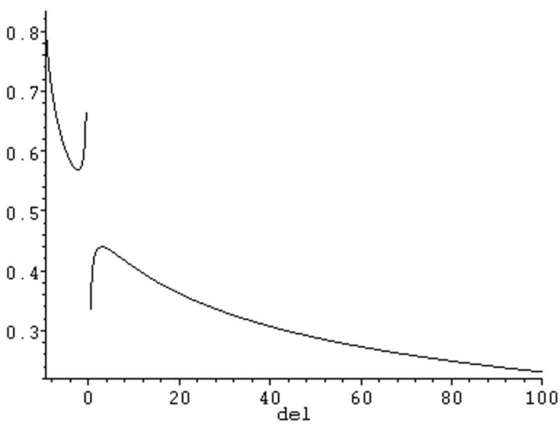

FIGURE 1 Behavior of *lb*/*ub* as a function of δ, for the case in which $\gamma = .81$ and $\sigma^2_{X_+|T} = 10$.

graph, that is, *lb*, decreases on (–10, –2.56), and, at –2.56, has a minimum of .57. Thus, when $\sigma^2_{X_+|T'} - \sigma^2_{X_+|T} = -2.56$, any $\pi_T > .57$ will produce a 1-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$, and any $\pi_T < .57$ will produce a 2-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$. The curve on the right, that is, *ub*, increases on (0, 2.56), and has a maximum of .44 at 2.56. It then decreases on (2.56, ∞), converging to zero in the limit. Thus, when $\delta = 2.56$, any $\pi_T < .44$ will produce a 1-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$, and any $\pi_T > .44$ will produce a 2-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$.

Figure 2 summarizes the conditional covariance functions yielded by the various subclasses of T-structures as $(p-2)$ becomes large. As is clear from Figure 2:

1. TNUV-structures for which $\delta < 0$ and $\pi_T > lb > .5$ yield a single-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$. Such structures include those for which: (a) π_T is very large; and (b) $\pi_T > .5$, γ^2 is small (poor indicators) and $\delta \approx -\delta^*$.
2. TNUV-structures for which $\delta < 0$ and $\pi_T < lb$ yield a two-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$. Such structures include those for which: (a) $\pi_T < .5$; and (b) π_T assumes virtually any value, γ^2 is large, and δ is either small or large negative.
3. TNUV-structures for which $\delta > 0$ and $\pi_T > ub$ yield a two-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$. Such structures include those for which: (a) $\pi_T > .5$; and (b) π_T assumes virtually any value, γ^2 is large, and δ is either small or large positive.
4. TNUV structures for which $\delta > 0$ and $\pi_T < ub < .5$ yield a single-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$. Such structures include those for which: (a) π_T is very small; and (b) $\pi_T < .5$, γ^2 is small (poor indicators) and $\delta \approx \delta^*$.

EXAMPLES

Table 1 provides *ub*(*lb*), *v*, h_1, h_3, and the peakedness of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ yielded by twelve TNUV-structures (particular choices of $\left[\pi_T, \mu_{X_+|T}, \mu_{X_+|T'}, \sigma^2_{X_+|T}, \sigma^2_{X_+|T'}\right]$). The final four scenarios are taken from Meehl and Golden (1982). For structures 1, 3, and 5, $\sigma^2_{X_+|T'} > \sigma^2_{X_+|T}$, so that $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ is a concave function, while for structures 2, 4, and 6, $\sigma^2_{X_+|T'} < \sigma^2_{X_+|T}$, so that $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ is convex. The $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ of structures 1–6 are displayed in Figure 3. For structures 1, 3, and 5, $\pi_T < ub$, while for structures 2, 4, and 6, $\pi_T > lb$. Hence, all of these structures yield a $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ that does not cross .5, and, hence, a single-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$.

For structure 7, $\pi_T = .44$, $\mu_{X_+|T} = 4, \mu_{X_+|T'} = 1, \sigma^2_{X_+|T} = 1$, and $\sigma^2_{X_+|T'} = 3$, so that $\sigma^2_{X_+|T'} > \sigma^2_{X_+|T}$, and $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ is concave. For this structure, $\gamma^2 = 9$, $\delta = 2$, and

$$\delta^* = \sqrt{\gamma^2 \sigma^2_{X_+|T}} = 5.2,$$

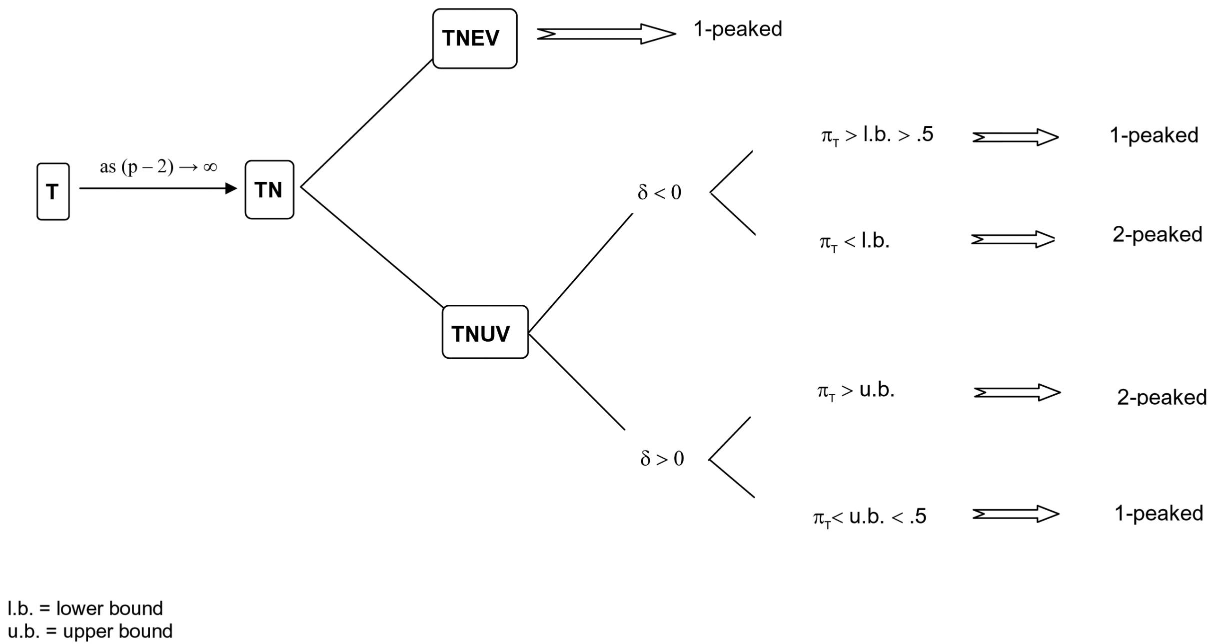

FIGURE 2 Peakedness of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ yielded by T-structures as $(p-2) \to \infty$.

TABLE 1
lb(*ub*), the Locations of the Maxima(minima), h_1, *v*, and h_3, of $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ and $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$, and the Number of Peaks of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ Under Taxonic Structures With Various Combinations of $\pi_T, \mu_{X_+|T}, \mu_{X_+|T'}, \sigma^2_{X_+|T}$, and $\sigma^2_{X_+|T'}$

	π_T	$\mu_{X_+\|T}$	$\mu_{X_+\|T'}$	$\sigma^2_{X_+\|T}$	$\sigma^2_{X_+\|T'}$	*ub(lb)*	h_1	*v*	h_3	$C(X_1, X_2\|X_+ = h)$
1	.05	2	1	1	3	.31		2.5		1-peaked
2	.95	2	1	2	1	.70		0		1-peaked
3	.05	3	1	1	3	.175		4		1-peaked
4	.71	2	1	2	1	.70		0		1-peaked
5	.14	3	1	1	10	.2		2.11		1-peaked
6	.87	2	1	20	1	.82		.95		1-peaked
7	.44	4	1	1	3	.057	2.73	5.5	8.27	2-peaked
8	.95	6	1	2	1	≈ 1	2.29	–4	–10.29	2-peaked
MG1	.5	12	8	4.41	3.61	.99	10	–10	–30	2-peaked
MG2	.5	12	8	5.29	2.89	.974	10	3.18	–3.62	2-peaked
MG3	.5	12	8	6.25	2.25	.925	9.95	5.75	1.55	2-peaked
MG4	.5	12	8	9	1	.89	9.67	7.5	5.32	2-peaked

Note. MG1-MG4 are from the Monte Carlo study of Meehl and Golden (1982).

and the combination of good indicators and a δ that is not close to δ^* produces a *ub* that is equal to .057. Thus, it would take a value of π_T that is less than .057 before $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ failed to cross .5, and, because $\pi_T = .44$, $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ crosses .5, and $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is 2-peaked. The indicators of structure 8 are even better than those of structure 7, and the *lb* it yields is essentially unity. As a result, even given the relative extremity of π_T (.95), the $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ it yields still crosses .5, and the $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ it yields is 2-peaked.

For each of the four Meehl and Golden (MG) structures, $\sigma^2_{T'} < \sigma^2_T$, $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ is convex, and the resulting *lb* is extreme (but decreases as δ decreases over these scenarios). Figure 4 displays $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ for each of these structures. For each of these structures, $\pi_T = .5$, $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ must then cross .5, and $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is 2-peaked. Figure 5 displays $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ and $f_{X_+}(h)$ for MG1 and MG4. For MG1, $h_1 = 10$ and $h_3 = -30$, and $f_{X_+}(h)$ assumes non-zero values for values of *h* that lie between 0 and 20. Hence, it is unlikely that the peak of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ located at –30 would be revealed in empirical conditional covariance plots. For MG4, on the other hand, $h_1 = 9.67$, $h_3 = 5.32$, $f_{X_+}(h)$ assumes non-zero values at both of these peaks, and the 2-peakedness of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ would be more likely to appear in empirical plots.

DISCUSSION

Paul Meehl was an empirical realist. To many empirical realists, latent structures are not merely casually "assumed" in order to facilitate data analysis, but are "causal features of natural reality generally concealed from perception but

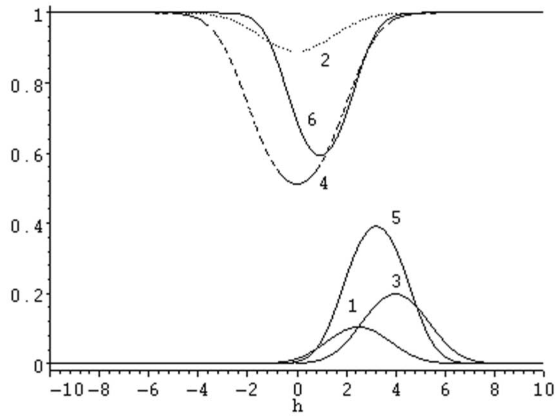

FIGURE 3 Behavior of $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ under various combinations of π_T, $\mu_{X_+|T}, \mu_{X_+|T'}, \sigma^2_{X_+|T}$, and $\sigma^2_{X_+|T'}$.

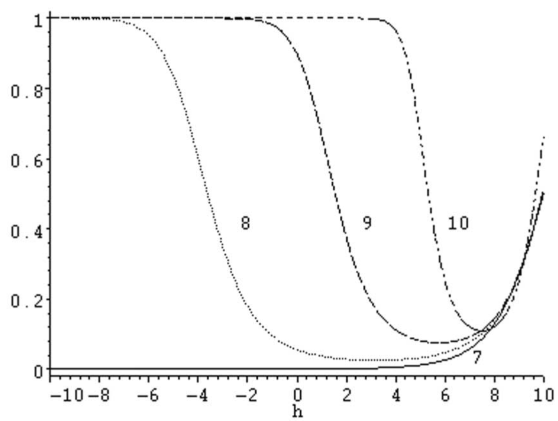

FIGURE 4 Behavior of $P(\boldsymbol{\theta} = T|\mathbf{X}_+ = h)$ for Meehl and Golden (1982) scenarios 1 to 4.

knowable through their data consequences" (Rozeboom, 1984, p. 212). To researchers who share Meehl's perspective, latent variable models are not merely tools by which data can be described and reduced, but, rather, tools employed to detect and study existing, but unobservable (latent), structures. Meehl has suggested in many articles that taxa (true, "natural kinds" or types) occur in nature, and that the existence of such taxa (and their complement classes) is the cause of the phenomena observed within particular domains of psychological investigation. According to Meehl, a chief task of the psychological scientist is to detect and study such taxa when, in fact, they do underlie domains of observable phenomena.

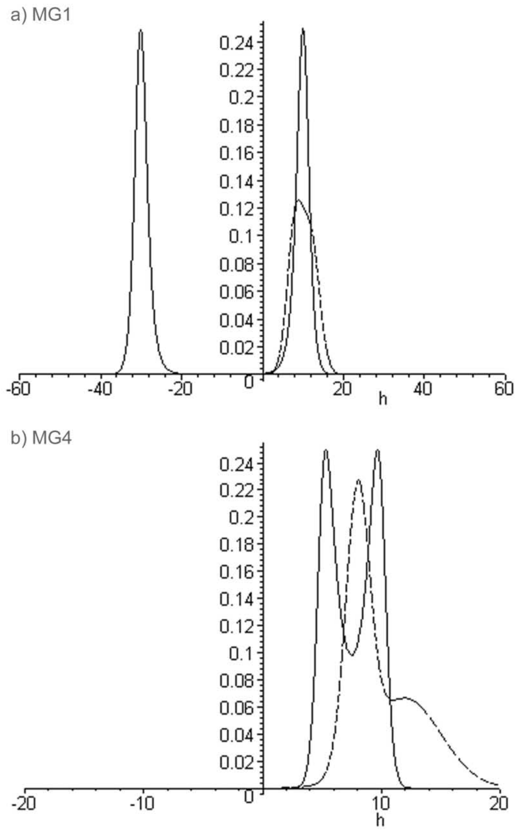

FIGURE 5 $f_{X_+}(h)$ and $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ for Meehl and Golden (1982) scenarios 1 and 4.
Note. Solid line = $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$; broken line = $f_{X_+}(h)$.

The use of latent variable technology in the detection of a particular latent structure, *S*, requires knowledge of the observed (manifest) properties that are implied by *S*, that is, properties that are necessary conditions of *S*, and, also, observed properties that imply *S*, that is, sufficient conditions of *S*. If observable property *t* is a necessary condition of *S*, then, when ~*t* is the case, it may validly be concluded that *S* does not underlie the data. If *t* is merely a necessary condition of *S*, then, of course, it is not valid to conclude that, when *t* is the case, *S* does underlie the data. If, on the other hand, *t* is a sufficient condition for *S*, then, when *t* is the case, it may validly be concluded that *S* underlies the data. If t is merely a sufficient condition of *S*, then, when ~*t* is the case, it is not valid to conclude that *S* does not underlie the data. Obviously, it is desirable that given property *t* be both a necessary and sufficient condition of S, for then a valid decision about S can be made in each of the cases t and ~t. It is well known that if $\mathbf{\Sigma} \neq \mathbf{\Lambda\Lambda}' + \mathbf{\Psi}$, in which $\mathbf{\Lambda}$ is a p by r matrix, $r < p$, and $\mathbf{\Psi}$ is diagonal and positive definite, then the data did not arise from an r-dimensional linear factor structure. On the other hand, evidence that $\mathbf{\Sigma} = \mathbf{\Lambda\Lambda}' + \mathbf{\Psi}$ is not evidence that the data arose from an r-dimensional linear factor structure, because $\mathbf{\Sigma} = \mathbf{\Lambda\Lambda}' + \mathbf{\Psi}$ is not a sufficient condition of this structure. In fact, the unidimensional r-degree polynomial factor structure yields precisely the same covariance structure (McDonald, 1967).

As Meehl has argued on many occasions, a consideration of the necessary and sufficient properties of T-structures makes it clear that conventional latent variable technologies in which the latent variable is continuously distributed (e.g., linear factor analysis) cannot be used in coherent attempts to detect T-structures. This observation was the motivation for his development of his taxometric procedures, of which MAXCOV is but one example. The claim at the root of MAXCOV is that single-peakedness of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is a necessary condition of the T-structure. If this were true, evidence, in a given empirical context, that $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ was not single-peaked could rightly be taken as evidence that the data did not arise from a T-structure. It remains unknown whether the single-peakedness of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is a necessary condition of T-structures. However, it been shown in this article that, if $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ *is* a necessary condition of T-structures, this necessity does not result from the circumstances described by Meehl's Hypothesis, because Meehl's Hypothesis is false.

On the other hand, it has been established that, as $(p - 2)$, the number of indicators in the conditioning set, becomes large, T-structures necessarily yield either single-, or two-peaked, $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$, depending on the values of the parameters $\left[\pi_T, \mu_{X_+|T}, \mu_{X_+|T'}, \sigma^2_{X_+|T}, \sigma^2_{X_+|T'}\right]$. Thus, the researcher who employs a number of indicator variates in the conditioning set can rightly take evidence that $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is not peaked (either single- or two-peaked) as evidence that the data did not arise from a T-structure. On the other hand, because, under Meehl's Hypothesis, the (single) maximum of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is taken as revealing the hitmax cut point, namely that point $h_{\max}$ at which $P(\boldsymbol{\theta} = T\,|\mathbf{X}_+ = h_{\max}) = P(\boldsymbol{\theta} = T'|\mathbf{X}_+ = h_{\max})$, the existence of T-structures that yield a 2-peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ obviously poses problems for empirical applications of MAXCOV. As it stands, it is not known whether the peakedness of $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ is sufficient for T-structures, because it is not known whether there exists any other class of latent structures for continuous indicators that yield peaked $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$.

Meehl has also claimed that C-structures yield flat $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$. If this were true, then evidence, in a given context, that $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ was not flat could rightly be taken as evidence that the data did not arise from a C-structure. It would, then, follow that evidence that $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$ was peaked (single or otherwise) would eliminate all C-structures as possible origins of the data. It is unclear whether or not this claim of Meehl's is true. Certainly, for the case of dichotomous indicators, there exist C-structures (e.g., certain Rasch structures, see Maraun et al., 2003) that yield non-flat $C(\mathbf{X}_1, \mathbf{X}_2|\mathbf{X}_+ = h)$.

It might be asked why any attention should be paid to decision-making machinery of the type that is MAXCOV, when there are available to the researcher sophisticated likelihood based inferential techniques for latent class and profile modelling (see, e.g., Magidson & Vermunt, 2004), these techniques apparently offering the researcher the added advantage of not having to restrict himself to the case of two classes. The making of valid decisions about whether data arose from a particular latent structure, *S*, rests on both population level and inferential considerations. In the first place, it must be established at the population level (non-inferentially) that there do exist manifest properties that can be employed to make valid decisions. This is the task of deducing necessary and/or sufficient conditions of *S*. Following Guttman (e.g., 1977) and Meehl, it is our belief that such population issues must be resolved before inferential issues can be fruitfully addressed. At present, little is known about the population-level basis for distinguishing between T- and C-structures, and T- and multi-class discrete structures. We, therefore, believe that the proliferation of likelihood-based inferential procedures "for fitting latent class and profile models" can only lead to a proliferation of empirical claims whose logical standing is unclear. While it is, of course, attractive to envision possessing "the flexibility of fitting a 2 to k class latent profile structure," it might be asked whether one should not first know what *are* the manifest properties on the basis of which the researcher can validly claim to have detected and distinguished between these various latent structures (if, in fact, such properties do exist). Meehl's taxometric program represents an attempt to provide such answers with respect one particular class of latent structures, the taxonic structures.

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APPENDIX

Let the 2 by 1 vector $\underline{\mathbf{X}}$ contain the random variates $\mathbf{X}_1$ and $\mathbf{X}_2$. Then, by definition, the 2 by 2 covariance matrix of $\mathbf{X}_1$ and $\mathbf{X}_2$ conditional on $\mathbf{X}_+ = h$, is

$$C\left[\left(\mathbf{X}_1,\mathbf{X}_2\right)\middle|\mathbf{X}_+ = h\right] = \int_A \underline{\mathbf{X}}\underline{\mathbf{X}}' f_{\underline{\mathrm{X}}|\mathrm{X}_+=h}\, d\underline{\mathrm{X}} - \underline{\mu}_{\underline{\mathrm{X}}|\mathrm{X}_+=h}\underline{\mu}_{\underline{\mathrm{X}}|\mathrm{X}_+=h}, \tag{35}$$

in which A is the range space of random vector $\underline{\mathbf{X}}$. Because

$$f_{\underline{X}|X_+=h}=\frac{\sum_t^{(T',T)} f_{\underline{X}|\theta=t,X_+=h}f_{X_+=h|\theta=t}P(\theta=t)}{f_{X_+}(h)}=\sum_t^{(T',T)} f_{\underline{X}|\theta=t,X_+=h}P\big(\theta=t\big|X_+=h\big),\quad (36)$$

Equation 35 can be rewritten as

$$C\big[(\mathbf{X}_1,\mathbf{X}_2)\big|\mathbf{X}_+=h\big]=\int_A \underline{\mathbf{X}\mathbf{X}'}\sum_t^{(T',T)} f_{\underline{X}|\psi=t,X_+=h}P\big(\psi=t\big|X_+=h\big)d\underline{X}-\underline{\mu}_{\underline{X}|X_+=h}\underline{\mu}'_{\underline{X}|X_+=h}$$

$$=\sum_t^{(T',T)}\int_{\mathbf{A}}\underline{\mathbf{X}\mathbf{X}'}f_{\underline{X}|\psi=t,X_+=h}d\underline{X}P\big(\psi=t\big|X_+=h\big)-\underline{\mu}_{\underline{X}|X_+=h}\underline{\mu}'_{\underline{X}|X_+=h},\quad (37)$$

which, using the definitions provided above Equation 4, can be rewritten as

$$(1-\pi_{Th})\big(\Phi_{T'h}+\underline{\mu}_{T'h}\underline{\mu}'_{T''h}\big)+\pi_{Th}\left(\Phi_{Th}+\underline{\mu}_{Th}\underline{\mu}'_{Th}\right)-\underline{\mu}_{\underline{X}|X_+=h}\underline{\mu}'_{\underline{X}|X_+=h}\quad (38)$$

Note the identity $\underline{\mu}_{\underline{X}|X_+=h}=(1-\pi_{Th})\underline{\mu}_{T'h}+\pi_{Th}\underline{\mu}_{Th}$, substitute the right member into Equation 38, simplify, and conditional covariance structure 4 follows.