**Differential Relations for Fluid Flow**

In this approach, we apply our four basic conservation laws to an infinitesimally small control volume. The differential approach provides point-by-point details of a flow pattern as oppose to control volume technique that provide gross-average information about the flow.

**Acceleration field of a fluid**

The Cartesian vector form of a velocity filed can be written as:

\[ V(r,t) = i \cdot u(x,y,z,t) + j \cdot v(x,y,z,t) + k \cdot w(x,y,z,t) \]

The flow filed is the most important variable in the fluid mechanics, i.e., knowledge of the velocity vector field is equivalent to solving a fluid flow problem.

The acceleration vector field can be calculated:

\[
\frac{du(x,y,z,t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t}
\]

\[
= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}
\]

\[
= \frac{\partial u}{\partial t} + (V \nabla)u
\]

where the compact dot products is:

\[ u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} = V \cdot \nabla \quad and \quad \nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \]

With a similar approach, we obtain the total acceleration vector:

\[ a = \frac{dV}{dt} = \frac{\partial V}{\partial t} + \left( u \frac{\partial V}{\partial x} + v \frac{\partial V}{\partial y} + w \frac{\partial V}{\partial z} \right) \]

\[ = \frac{\partial V}{\partial t} + (V \cdot \nabla)V \]

The term \( \frac{\partial V}{\partial t} \) is called the **local acceleration** and vanishes if the flow is steady. The three terms in the parentheses are called the **convective acceleration** and rises when the particles move through regions of spatially varying velocity, e.g. nozzle.

The total time derivative \( (d/dt) \) is sometimes called the **substantial** or **material derivative** can be applied any variable such as pressure. This operator sometimes assigned a special symbol D/Dt.

**The differential equation of mass conservation**

All basic equations can be derived by considering an elemental system. Figure 1 shows the control volume \( (dx, dy, dz) \) in which flow through each side of the element is approximately one-dimensional. Since the size of the element is so small, we can assume that all the fluid properties are uniform and constant within the element.
The conservation of mass for the element can be written as:

\[
\frac{\partial \rho}{\partial t} dx \, dy \, dz + \frac{\partial}{\partial x} (\rho u) dx \, dy \, dz + \frac{\partial}{\partial y} (\rho v) dy \, dz + \frac{\partial}{\partial z} (\rho w) dx \, dy \, dz = 0
\]

After dividing by the volume of the element:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0
\]

The continuity equation for the cylindrical polar coordinates is:

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0
\]

where velocity vector \( V = (v_r, v_\theta, v_z) \).

For steady compressible flow, continuity equation simplifies to:
\[ \begin{cases} \text{Cartesian,} & \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0 \\ \text{Cylindrical,} & \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0 \end{cases} \]

For incompressible flow, continuity equation can be further simplified since density changes are negligible:

\[
\nabla \cdot \mathbf{V} = 0
\]

\[
\begin{cases} \text{Cartesian,} & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\ \text{Cylindrical,} & \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (v_\theta) + \frac{\partial}{\partial z} (v_z) = 0 \end{cases} \]

A flow can be considered incompressible when \( Ma \leq 0.3 \).

Note: the continuity equation is always important and must always be satisfied for a rational analysis of a flow pattern.

**The differential equation of linear momentum**

In a Cartesian coordinates, the momentum equation can be written as:

\[
\sum F = \rho \frac{dV}{dt} dx \, dy \, dz
\]

There are types of forces: body forces and surface forces.

**Body forces** are due to external fields such as gravity and magnetism fields. We only consider gravity forces:

\[
dF_{grav} = \rho \mathbf{g} dx \, dy \, dz \quad \text{where} \quad \mathbf{g} = -g \mathbf{k}
\]

The **surface forces** are due to the stresses on the sides of the control surface. These stresses are the sum of hydrostatic pressure plus viscous stresses \( \tau_{ij} \) which arise from the motion of the fluid:

\[
\sigma_{ij} = \begin{bmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{bmatrix}
\]

Unlike velocity, stresses and strains are nine-component tensors and require two subscripts to define each component.

The net surface force due to stresses in the \( x \)-direction can be found as:

\[
dF_{x,surf} = \left[ \frac{\partial}{\partial x} (\sigma_{xx}) + \frac{\partial}{\partial y} (\sigma_{yx}) + \frac{\partial}{\partial z} (\sigma_{zx}) \right] dx \, dy \, dz
\]
Similarly we can find the net surface force in \( y \) and \( z \) direction. After summing them up and dividing through by the volume, we get:

\[
\left( \frac{dF}{dxdydz} \right)_{viscous} = i \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + j \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + k \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right)
\]

\[
= \nabla \cdot \tau_{ij}
\]

Note: the surface force is the sum of the pressure gradient and the divergence of the viscous stress tensor.

Therefore the linear momentum equation for an infinitesimal element becomes:

\[
\rho g - \nabla p + \nabla \cdot \tau_{ij} = \rho \frac{dV}{dt}
\]

where

\[
\frac{dV}{dt} = \frac{\partial V}{\partial t} + \left( u \frac{\partial V}{\partial x} + v \frac{\partial V}{\partial y} + w \frac{\partial V}{\partial z} \right)
\]

This is a vector equation, and can be written as:

\[
\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)
\]

\[
\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right)
\]

\[
\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)
\]
Special cases of momentum equation:

Inviscid flow: Euler’s equation
When the viscous terms are negligible, i.e. \( \tau_{ij} = 0 \)

\[
\rho g - \nabla p = \rho \frac{dV}{dt}
\]

Euler’s equation can be integrated along a streamline to yield the frictionless Bernoulli equation.

Newtonian fluid: Navier-Stokes equation
For a Newtonian fluid, the viscous stresses are proportional to the element strain rates and the coefficient of viscosity.

\[
\begin{align*}
\tau_{xx} &= 2\mu \frac{\partial u}{\partial x} \\
\tau_{yy} &= 2\mu \frac{\partial v}{\partial y} \\
\tau_{zz} &= 2\mu \frac{\partial w}{\partial z} \\
\tau_{xy} &= \tau_{yx} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\tau_{xz} &= \tau_{zx} = \mu \left( \frac{\partial w}{\partial y} + \frac{\partial \bar{u}}{\partial z} \right) \\
\tau_{yz} &= \tau_{zy} = \mu \left( \frac{\partial w}{\partial z} + \frac{\partial \bar{v}}{\partial x} \right)
\end{align*}
\]

where \( \mu \) is the viscosity coefficient. Substituting shear stresses in the momentum equation, for a Newtonian fluid with constant density and viscosity, we get:

\[
\begin{align*}
\rho g_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) &= \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \\
\rho g_y - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) &= \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \\
\rho g_z - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) &= \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)
\end{align*}
\]

These are the incompressible flow Navier-Stokes equations.

Note: Navier-Stokes equations have four unknowns: \( \rho, u, v, \) and \( w \). They should be combined with the continuity equation to form four equations for these unknowns.

Navier-Stokes equations have a limited number of analytical solutions; these equations typically are solved numerically using computational fluid dynamics (CFD) software and techniques.

The differential equation of angular momentum
Application of the integral theorem to a differential element gives that the shear stresses are symmetric:

\[
\tau_{ij} = \tau_{ji}
\]
Therefore, there is no differential angular momentum equation.

**The boundary conditions for the basic equations**

We have 3 differential equations to solve: i) continuity equation, ii) momentum, and iii) energy. Typically, the density is variable, so the three equations contain 5 unknowns: $p, V, p, \dot{u}$ and $T$. Therefore, we need 2 additional relations to complete the system of equations. These are provided by data or algebraic expressions for state relations of thermodynamic properties such as ideal gas equation of state:

$$\rho = \rho(p, T) \quad \text{and} \quad \dot{u} = \dot{u}(p, T)$$

For ideal gas, we have: $\rho = p/RT$ and $\dot{u} = c_v T$. So, we need to set proper, initial and boundary conditions for each variable.

Some important boundary conditions:

- At solid wall: $V_{\text{fluid}} = V_{\text{wall}}$ (no-slip condition) \hspace{1cm} $T_{\text{fluid}} = T_{\text{wall}}$ (no-temperature jump)
- At inlet or outlet section of the flow: $V, p, T$ are known
- At a liquid-gas interface: equality of vertical velocity across the interface (kinematic boundary condition).
- Mechanical equilibrium at liquid-gas interface, $(\tau_{zx})_{liq} = (\tau_{zx})_{gas}$ and $(\tau_{zy})_{liq} = (\tau_{zy})_{gas}$
- At a liquid-gas interface: heat transfer must be the same, $(q_z)_{liq} = (q_z)_{gas}$, or

$$\left(k \frac{\partial T}{\partial z}\right)_{liq} = \left(k \frac{\partial T}{\partial z}\right)_{gas}$$
Incompressible flow with constant properties
Flow with constant $\rho, \mu, and k$ is a basic simplification that is very common in engineering problem that leads to:

Continuity
$$ \nabla V = 0 $$
Momentum
$$ \rho \frac{dV}{dt} = \rho g - \nabla p + \mu \nabla^2 V $$

For frictionless or inviscid flows in which $\mu = 0$. The momentum equation reduces to Euler’s equation:
$$ \rho \frac{dV}{dt} = \rho g - \nabla p $$

Some illustrative incompressible viscous flows
Couette flow between a fixed and a moving plate

Consider two-dimensional incompressible plane viscous flow between parallel plates, $\frac{\partial}{\partial z} = 0$, a distance $2h$ apart, as shown in Fig. 4. We assume that the plates are very wide and very long and that the flow is essentially axial.

$$ v = w = 0 \text{ and } u \neq 0 $$

![Fig. 4: Incompressible viscous flow between parallel plates, a) no pressure gradient; b) pressure gradient with both plates fixed.](image)

From continuity equation, we learn:

$$ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \rightarrow \frac{\partial u}{\partial x} = 0 \text{ or } u = u(y) \text{ only} $$

We neglect gravity effects and assume fully-developed flow. Substitute $u = u(y)$ in the Navier-Stokes equation for $x$-direction:
\[
\rho g_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)
\]

After simplifications, it reduces to:

\[
\frac{d^2 u}{dy^2} = 0 \text{ or } u = C_1 y + C_2
\]

We need two boundary conditions to find the constants \(C_1\) and \(C_2\).

At \(y = +h\), \(u = V = C_1 h + C_2\)

At \(y = -h\), \(u = 0 = C_1 (-h) + C_2\)

Therefore, the solution for flow between plates with a moving upper wall is:

\[
u = \frac{V}{2h} y + \frac{V}{2} \quad -h \leq y \leq +h
\]

This is called **Couette flow due to a moving wall**: a linear velocity profile with no slip at each wall.

**Couette flow due to pressure gradient between two fixed plates**

Let’s consider case (b) in Fig. 4, where both plates are fixed but pressure varies in the \(x\)-direction. If \(v = w = 0\), the continuity equation lead to the same result, namely:

\[
u = u(y)
\]

The \(x\)-momentum equation changes because the pressure is variable:

\[
\rho g_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)
\]

\[
\mu \frac{d^2 u}{dy^2} = \frac{\partial p}{\partial x}
\]

Also, since \(v = w = 0\) and gravity is neglected, the \(y\) and \(z\) momentum equations lead to:

\[
\frac{\partial p}{\partial y} = 0 \text{ and } \frac{\partial p}{\partial z} = 0 \rightarrow p = p(x) \text{ only}
\]

Thus, the pressure gradient in \(x\)-direction is the only and total gradient:

\[
\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} = \text{Const} < 0
\]

The solution is accomplished by double integration:
\[ u = \frac{1}{\mu} \left( \frac{dp}{dx} \right) \frac{y^2}{2} + C_1 y + C_2 \]

The constants can be found from the no-slip boundary condition at the walls:

at \( y = \pm h \): \( u = 0 \) Thus, \( C_1 = 0 \) and \( C_2 = -\frac{dp}{dx} \frac{h^2}{2\mu} \)

The velocity distribution in a channel due to pressure gradient is:

\[ u = -\frac{h^2}{2\mu} \frac{dp}{dx} \left( 1 - \frac{y^2}{h^2} \right) \]

The flow forms a Poiseuille parabola of constant negative curvature, where the maximum velocity occurs at the center.

**Fully-developed laminar pipe flow**

This is one of the most useful exact solutions to the Navier-Stokes equation: *fully-developed incompressible flow in straight circular pipe of radius \( R \).*

**Fully-developed flow:** refers to the flow in a region far enough from the entrance that the flow is purely axial. As a result, the velocity distribution in the tube is fixed (not changing along the tube).

Neglecting gravity effects and assuming axial symmetry, i.e., \( \frac{\partial}{\partial \theta} = 0 \) and \( v_\theta = 0 \), the continuity equation in cylindrical coordinates, reduces to:

\[ \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (v_\theta) + \frac{\partial}{\partial z} (v_z) = 0 \rightarrow v_z = v_z(r) \]

It means that the flow proceeds straight down the pipe without radial motion. The \( r \)-momentum equation in cylindrical coordinates, simplifies to \( \frac{\partial p}{\partial r} = 0 \) or \( p = p(z) \) only. The \( z \)-momentum equation in cylindrical coordinates, reduces to:

\[ \rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_\theta \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z \]

\[ \frac{\mu}{r} \frac{d}{dr} \left( r \frac{dv_z}{dr} \right) = \frac{dp}{dz} = Const < 0 \]

This equation is linear and can be integrated twice:

\[ v_z = \frac{dp}{dz} \frac{r^2}{4\mu} + C_1 \ln(r) + C_2 \]

Applying, the no-slip boundary condition at the wall and finite velocity at the centerline, gives us:
No-slip at \( r = R \): \( v_z = 0 = \frac{dp}{dz} R^2 + C_1 \ln(R) + C_2 \)

Finite velocity at \( r = 0 \): \( v_z = \text{finite} = 0 + C_1 \ln(0) + C_2 \to C_1 = 0 \)

The final solution for fully-developed Hagen-Poiseuille flow is:

\[ v_z = \left( -\frac{dp}{dz} \right) \frac{1}{4\mu} (R^2 - r^2) \]

The velocity profile is a paraboloid with a maximum at the centerline.

**Flow between long concentric cylinders**

Consider an incompressible flow between two concentric cylinders, as shown in Fig. 5. There is no axial motion or end effect, \( v_z = \frac{\partial}{\partial z} = 0 \).

![Flow between long concentric cylinders](image)

Fig. 5: Incompressible viscous flow between long cylinders.

The continuity, with \( v_z = 0 \), reduces:

\[
\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (v_\theta) + \frac{\partial}{\partial z} (v_z) = 0 \to rv_r = \text{const.}
\]

Note that \( v_\theta \) does not vary with \( \theta \). Also since \( v_r = 0 \) at both the inner and outer cylinders, it follows that \( v_r = 0 \) everywhere, as a result, the motion will be purely circumferential, \( v_\theta = v_\theta(r) \). The momentum equation in \( \theta \) direction is:

\[
\rho (\nabla \cdot v) v_\theta + \frac{\rho v_\theta v_r}{r} = -\frac{1}{r} \frac{\partial}{\partial \theta} + \rho g_\theta + \mu \left( \nabla^2 v_\theta - \frac{v_\theta}{r^2} \right)
\]

In cylindrical coordinates, we have:
As a result, the momentum equation becomes:

$$0 = \mu \left( \nabla^2 v_\theta - \frac{v_\theta}{r^2} \right) \rightarrow \nabla^2 v_\theta = \frac{v_\theta}{r^2}$$

This is a linear 2\(^{nd}\) order ordinary differential equation:

$$v_\theta = C_1 r + \frac{C_2}{r}$$

The constants can be found by applying boundary conditions:

At \( r = r_0 \): \( v_\theta = 0 = C_1 r_0 + C_2 / r_0 \)

At \( r = r_i \): \( v_\theta = \Omega r_i = C_1 r_i + C_2 / r_i \)

The final solution for the velocity distribution is:

$$v_\theta = \frac{r_0 - r}{r - r_0} \Omega_i r_i \frac{r_0}{r_i}$$