The complete (small-slope) solution to this problem is:

$$\eta = h \exp[-(\rho g/Y)^{1/2} x], \quad \text{where } h = (Y/\rho g)^{1/2} \cot \theta$$  \hspace{1cm} \text{Ans.}$$

The formula clearly satisfies the requirement that $\eta = 0$ if $x = \infty$. It requires “small slope” and therefore the contact angle should be in the range $70^\circ < \theta < 110^\circ$.

---

**1.69** A solid cylindrical needle of diameter $d$, length $L$, and density $\rho_n$ may “float” on a liquid surface. Neglect buoyancy and assume a contact angle of $0^\circ$. Calculate the maximum diameter needle able to float on the surface.

**Solution:** The needle “dents” the surface downward and the surface tension forces are upward, as shown. If these tensions are nearly vertical, a vertical force balance gives:

$$\sum F_z = 0 = 2YL - \rho_n g \frac{\pi}{4} d^2 L, \quad \text{or: } \quad d_{\max} \approx \sqrt[3]{\frac{8Y}{\pi \rho g}} \hspace{1cm} \text{Ans. (a)}$$

(b) Calculate $d_{\max}$ for a steel needle (SG $\approx 7.84$) in water at $20^\circ C$. The formula becomes:

$$d_{\max} = \sqrt[3]{\frac{8Y}{\pi \rho g}} = \sqrt[3]{\frac{8(0.073 \text{ N/m})}{\pi(7.84 \times 998 \text{ kg/m}^3)(9.81 \text{ m/s}^2)}} \approx 0.00156 \text{ m} \approx 1.6 \text{ mm} \hspace{1cm} \text{Ans. (b)}$$

---

**1.70** Derive an expression for the capillary-height change $h$, as shown, for a fluid of surface tension $Y$ and contact angle $\theta$ between two parallel plates $W$ apart. Evaluate $h$ for water at $20^\circ C$ if $W = 0.5$ mm.

**Solution:** With $b$ the width of the plates into the paper, the capillary forces on each wall together balance the weight of water held above the reservoir free surface:

$$\rho g W h b = 2(Yb \cos \theta), \quad \text{or: } \quad h \approx \frac{2Y \cos \theta}{\rho g W} \hspace{1cm} \text{Ans.}$$
For water at 20°C, Y ≈ 0.0728 N/m, ρg ≈ 9790 N/m³, and θ ≈ 0°. Thus, for W = 0.5 mm,
\[
h = \frac{2(0.0728 \text{ N/m}) \cos 0°}{(9790 \text{ N/m}^3)(0.0005 \text{ m})} \approx 0.030 \text{ m} \approx 30 \text{ mm} \quad \text{Ans.}
\]

1.71* A soap bubble of diameter D₁ coalesces with another bubble of diameter D₂ to form a single bubble D₃ with the same amount of air. For an isothermal process, express D₃ as a function of D₁, D₂, pₐₚₗ, and surface tension Y.

**Solution:** The masses remain the same for an isothermal process of an ideal gas:

\[
m₁ + m₂ = ρ₁υ₁ + ρ₂υ₂ = m₃ = ρ₃υ₃,
\]

or:
\[
\left(\frac{p_a + 4Y/r_1}{RT}\right)\left(\frac{\pi}{6}D₁^3\right) + \left(\frac{p_a + 4Y/r_2}{RT}\right)\left(\frac{\pi}{6}D₂^3\right) = \left(\frac{p_a + 4Y/r₃}{RT}\right)\left(\frac{\pi}{6}D₃^3\right)
\]

The temperature cancels out, and we may clean up and rearrange as follows:
\[
p_aD₁^3 + 8YD₁^2 = \left(p_aD₂^3 + 8YD₂^2\right) + \left(p_aD₃^3 + 8YD₃^2\right) \quad \text{Ans.}
\]

This is a cubic polynomial with a known right hand side, to be solved for D₃.

1.72 Early mountaineers boiled water to estimate their altitude. If they reach the top and find that water boils at 84°C, approximately how high is the mountain?

**Solution:** From Table A-5 at 84°C, vapor pressure pᵥ ≈ 55.4 kPa. We may use this value to interpolate in the standard altitude, Table A-6, to estimate

\[
z \approx 4800 \text{ m} \quad \text{Ans.}
\]

1.73 A small submersible moves at velocity V in 20°C water at 2-m depth, where ambient pressure is 131 kPa. Its critical cavitation number is Ca ≈ 0.25. At what velocity will cavitation bubbles form? Will the body cavitate if V = 30 m/s and the water is cold (5°C)?

**Solution:** From Table A-5 at 20°C read pv = 2.337 kPa. By definition,

\[
Ca_{crit} = 0.25 = \frac{2(p_a - pᵥ)}{\rho V^2} = \frac{2(131000 - 2337)}{(998 \text{ kg/m}^3)V^2}, \quad \text{solve } V_{crit} \approx 32.1 \text{ m/s} \quad \text{Ans. (a)}
\]