

Analytical Model for the Effective Thermal Conductivity Matrix of Anisotropic Composites

A. Akbari
PhD Student
Mechatronic Systems Engineering
Simon Fraser University
aakbari@sfu.ca

E. Kjeang
Assistant Professor
Mechatronic Systems Engineering
Simon Fraser University
ekjeang@sfu.ca

M. Bahrami
Assistant Professor
Mechatronic Systems Engineering
Simon Fraser University
mbahrami@sfu.ca

ABSTRACT

The effective thermal conductivity of anisotropic composites is studied. Using a Maxwell unit cell, a new analytical solution is developed for anisotropic composites based on the self-consistent field concept. The matrix is considered to be composed of randomly distributed elliptical particles within a continuous medium. The particles are allowed to have various aspect ratios and arbitrary orientation relative to the coordinate system axes with respect to which the components of effective thermal conductivity tensor is evaluated. The temperature distribution is solved in the unit cell and averaged to obtain all the spatial components of the effective thermal conductivity matrix. The proposed solution yields the Maxwell's theory at the limit where the particles' aspect ratio approaches unity. The present solution can serve as a general model for anisotropic composites with dilute dispersion of particles.

NOMENCLATURE

a = Aspect ratio, coefficients of transformation
 A = Area, m^2
 \mathbf{b} = Closure variable, m
 c = Distance between the focuses of ellipse, m
 \mathbf{I} = Unit tensor
 k = Thermal conductivity, W/mK
 \mathbf{K} = Effective thermal conductivity tensor, W/mK
 L = Half of major diameter of outer ellipse, m
 \mathbf{M} = Anisotropy coefficients matrix
 \mathbf{n} = Normal vector
 \mathbf{q} = Heat flux, W/m^2
 T = Temperature, K
 V = Volume of representative elementary volume, m^3

w = Half of minor diameter of inner ellipse, m
 W = Half of minor diameter of outer ellipse, m
 X = Coordinate in $O_{X^1X^2}$, m
 x = x-coordinate, m
 y = y-coordinate, m

Greek symbols

ε = Volume fraction
 θ = Orientation angle of particle, rad
 ℓ = Half of major diameter of inner ellipse, m
 δ = Kronecker delta tensor
 κ = Thermal conductivity ratio
 η = First coordinate in elliptic cylindrical coordinate
 ψ = Second coordinate in elliptic cylindrical coordinate

Subscript and super script

β = β -pahse
 σ = σ -pahse
 $\beta\sigma$ = $\beta\sigma$ -interface
 eff = Effective

INTRODUCTION

Over the past century, enormous research efforts have been focused on the determination of thermal properties of heterogeneous materials due to the abundant applications in industry and engineering. Among those are catalytic reactors, packed beds [1], composite materials [2], thermal interface materials [3], particle-in-liquid suspensions [4], cracks in solids [5] and new artificial materials such as gas diffusion layer (GDL) in proton exchange membrane fuel cells (PEMFC) [6]. GDL plays an important role in the operation of PEM fuel cells in that it has to fulfil several demands at the same time such as reactant gas transport, draining out liquid

water, conducting electrons and heat, etc. Effective conductivity (EC) and effective thermal conductivity (ETC) are two important properties of GDL which strongly affect and limit heat and mass transport as well as reaction rate. Therefore, the accurate knowledge of these parameters is essential to improve the power density and efficiency of PEMFCs.

Since the early work of Maxwell, numerous researches have been done to predict the effective thermal conductivity of various porous materials using different approaches. Hamilton and Crosser derived a semi-analytical geometrical model for two component mixture with non-spherical inclusions [7]. The effect of particle shape was taken into account through an empirical parameter which was related to *sphericity*. Hadley considered the randomness of particles distribution to correct the effective thermal conductivity at low porosities [8]. Hadley's model was modified by Verma et. al. in terms of constituent phases to obtain a better agreement with experimental data [9]. Yu and Choi extended the Hamilton and Crosser's model for particle-in-liquid systems to account for ordered liquid layer surrounding the particles [4]. More recently, Felske derived an analytical solution for the ETC of composites with randomly distributed spherical inclusions and interfacial layer within the self-consistent field model [10]. Within the framework of the volume averaging method (VAM) [11], Nozad et. al. developed a closure problem for two phase systems by solving which the ETC tensor can be completely described [12]. However, this closure problem can only be solved numerically, for geometries of practical application [1, 11, 13].

Even though most natural systems and many processed materials of practical importance are anisotropic, only a few researches have been dedicated to model anisotropic ETC tensor. Using the VAM, Lux et. al. numerically solved the closure problem of conduction heat transfer to obtain the complete ETC tensor of wood fibers [14]. Based on variational principle of Hashin and Shtrikman, Lu et. al. obtained the ETC tensor for composites with randomly distributed aligned spheroidal inclusions [5]. Using the same approach, Shafiro and Kachanov derived anisotropic ETC of non-randomly oriented inclusions of ellipsoidal shapes [15]. Although this approach leads to a closed form for ETC tensor, it cannot be considered as a completely exact solution. This method considers a composite as a matrix in which inclusions of different phase is embedded. Each particle is subject to a far field temperature gradient and the disturbance caused by other particles is either neglected or estimated by Hasselman-Johanson's equivalent inclusion model [5]. The final expression is in the form of truncated virial expansion and ETC tensor is diagonal [5, 15].

In this study, based on the self-consistent field concept, an analytical solution describing the complete ETC tensor of anisotropic composites with elliptical inclusions is presented. This model can serve as a general 2D model for ETC of highly porous anisotropic materials such as GDL.

EFFECTIVE THERMAL CONDUCTIVITY TENSOR

The tensor nature of effective thermal conductivity of heterogeneous materials has been addressed by several authors [1, 11, 13]. Therefore, it is convenient to represent ETC tensor in its precise meaning, i.e. its components transform according

to the transformation law associate with a specific coordinate transformation group [16]. On one hand, the effective thermal conductivity tensor has a rigorous definition in the VAM in terms of a closure variable vector which is to be solved in some representative unit cell. On the other hand, it can be proved that the ETC obtained in the self-consistent field model by averaging the temperature field is identical to those given by the solution of closure problem in the VAM [11, 13]. Hence, we take advantage of the definition of ETC tensor of the VAM to associate it with a Cartesian transformation group. In the following the definition of ETC within the framework of the VAM is briefly explained.

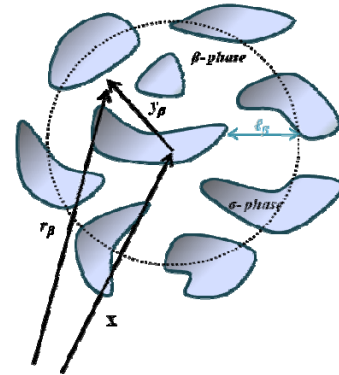


FIGURE 1. REPRESENTATIVE ELEMENTARY VOLUME

The principle of the analytical volume averaging process is applied by integrating the equation of heat conduction over each phase within a representative elementary volume (REV). Figure 1 shows an REV associated with centre point x , in a two phase porous medium with a microstructure of arbitrary geometry. Details regarding the selection and conditions of an REV are outlined by Whitaker [11].

The superficial average of a quantity defined in the phase is given by

$$\bar{\phi} = \frac{1}{V} \int_V \phi \, dV \quad (1)$$

The intrinsic average is defined as

$$\bar{\phi}^i = \frac{1}{V_i} \int_{V_i} \phi \, dV \quad (2)$$

One can decompose a local variable in the following manner

$$\phi = \bar{\phi} + \phi' \quad (3)$$

Using the above definitions and assuming local thermal equilibrium [11], one can drive the superficial average of heat flux vector in a two phase system shown in Fig. 1.

$$\bar{q} = \bar{q}^i + \bar{q}^s \quad (4)$$

where

$$\bar{q}^i = \frac{1}{V} \int_V q \, dV \quad (5)$$

$$\bar{q}^s = \frac{1}{V} \int_V q \, dV \quad (6)$$

In the above equations, ϕ^i is the volume fraction of each phase. The effective thermal conductivity tensor in Eq. (4) is defined as

$$\frac{\mathbf{K}_{eff}}{k_\beta} = (\varepsilon_\beta k_\beta + \varepsilon_\sigma k_\sigma) \mathbf{I} + \frac{k_\beta - k_\sigma}{V} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \mathbf{b}_\beta dA \quad (7)$$

where $A_{\beta\sigma}$ is the area of $\beta - \sigma$ interface, $\mathbf{n}_{\beta\sigma}$ is unit normal vector pointing from β -phase to σ -phase and \mathbf{b}_β is closure variable vector defined as

$$\tilde{T}_\beta = \mathbf{b}_\beta \cdot \nabla \langle T \rangle \quad (8)$$

The \mathbf{b}_β -field can be obtained through solving the heat conduction closure problem [11, 12] which is a boundary value problem in a unit cell with periodic boundary conditions. Without entering further into details of derivation of Eq. (4), (7) and associate closure problem, we refer the reader to [1, 11, 12] for comprehensive study.

We consider an infinite two dimensional matrix containing an elliptical inclusion with arbitrary orientation with respect to coordinate system of interest as shown in Fig. 2. Since the components of ETC tensor are dependent on the choice of coordinate system, it is to our advantage to represent the ETC associate with a transformation group. In this study, we are only concerned with the rectangular Cartesian transformation group.

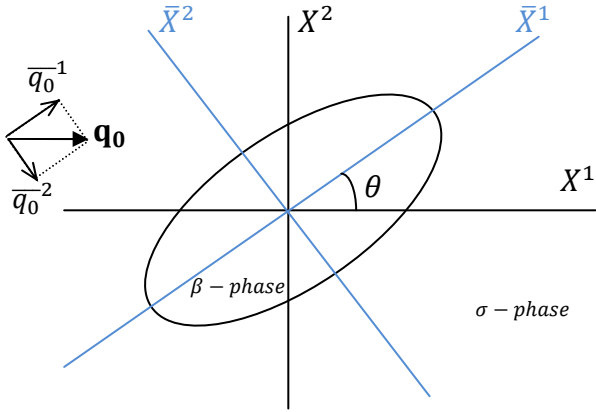


FIGURE 2. ELLIPTICAL INCLUSION IN MATRIX

In obtaining all the components of ETC for the composite shown in Fig. 2 with respect to $O_{X^1 X^2}$ coordinate system, the properties of rectangular Cartesian transformation are exploited. We take advantage of Eq. (8) to show how ETC components satisfy the transformation law. Since the left hand side of Eq. (8) is a scalar, so is the right hand side. This means that both sides of Eq. (8) must be invariant with respect to rectangular Cartesian transformation group. Given the fact that dot product of two vectors is invariant [16] and $\nabla \langle T \rangle$ is a vector, one can conclude that the components of \mathbf{b}_β satisfy the transformation law. The transformation law for the coordinate systems shown in Fig. 2 is given by [16]

$$X^j = a_k^j \bar{X}^k \quad (9)$$

A tensor notation is used throughout ($j = 1, 2$). Since translation of the origin of coordinate system has no effect on the tensor components, only rotation is considered.

$$a_k^j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad (10)$$

Equation (7) can be rewritten in components form as the following

$$\frac{K_{eff}^{ij}}{k_\beta} = (\varepsilon_\beta k_\beta + \varepsilon_\sigma k_\sigma) \delta^{ij} + \frac{k_\beta - k_\sigma}{V} \int_{A_{\beta\sigma}} n_{\beta\sigma}^i b_\beta^j dA \quad (11)$$

where δ^{ij} is Kronecker delta tensor. In writing Eq. (11), it is assumed that the ETC is an associated metric tensor of contravariant order 2. Note that it does not matter which tensor type is attributed to ETC in that the components of fundamental metric, associated metric and mixed tensors are identical with respect to rectangular Cartesian transformation group [16]. However, one should be careful about the tensor type when dealing with general coordinate transformations.

One can write Eq. (11) as associated with the coordinate transformation shown in Fig. 2.

$$\frac{K_{eff}^{ij}}{k_\beta} = (\varepsilon_\beta k_\beta + \varepsilon_\sigma k_\sigma) a_p^i a_q^j \bar{\delta}^{pq} + a_p^i a_q^j \left[\frac{k_\beta - k_\sigma}{V} \int_{A_{\beta\sigma}} \bar{n}_{\beta\sigma}^p \bar{b}_\beta^q dA \right] \quad (12)$$

which implies that

$$\frac{\bar{K}_{eff}^{pq}}{k_\beta} = (\varepsilon_\beta k_\beta + \varepsilon_\sigma k_\sigma) \bar{\delta}^{pq} + \frac{k_\beta - k_\sigma}{V} \int_{A_{\beta\sigma}} \bar{n}_{\beta\sigma}^p \bar{b}_\beta^q dA \quad (13)$$

and

$$K_{eff}^{ij} = a_p^i a_q^j \bar{K}_{eff}^{pq} \quad (14)$$

Equation (14) illustrates the tensor nature of ETC and how its components in two coordinate systems are related.

ANALYSIS

In general, for inclusions with arbitrary orientation, such as the one shown in Fig. 2, all the components of ETC with respect to $O_{X^1 X^2}$ are non-zero. This problem can be simplified by transforming $O_{X^1 X^2}$ into a coordinate system axes of which are along the principle axes of ETC tensor that is $O_{\bar{X}^1 \bar{X}^2}$ (see Fig. 2). In the case of ellipse, the principle axes of ETC are along the major and minor diameters. Based on the definition of \mathbf{K}_{eff} given by Eq. (7), for the elliptical inclusion shown in Fig. 2 it can be proved that ETC is a diagonal tensor with respect to $O_{\bar{X}^1 \bar{X}^2}$ due to the geometrical symmetry of ellipse and skew-symmetric \mathbf{b}_β [11]. It should also be mentioned that \bar{q}_0^2 makes no contribution to \bar{K}_{eff}^{11} and \bar{q}_0^1 makes no contribution to \bar{K}_{eff}^{22} . Therefore, one can take advantage of superposition principle to separately obtain the two principle components of ETC.

Effective thermal conductivity in x-direction

In this section an analytical solution for the first principle component of ETC with respect to $O_{\bar{X}^1 \bar{X}^2}$ for the two phase system shown in Fig. 2 is developed. Therefore, in this case, the heat flow direction is considered to be along the major axis of the ellipse. We use $x = \bar{X}^1$ and $y = \bar{X}^2$ to simplify the notation. Figure 3 illustrates the geometry considered for this case.

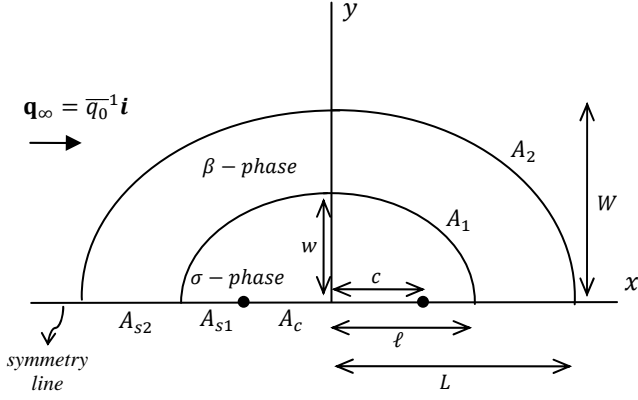


FIGURE 3. GEOMETRY FOR ETC IN X-DIRECTION

A uniform far field heat flux $\mathbf{q}_\infty = \bar{q}_0^{-1} \mathbf{i}$ is imposed in the medium at large distance from the inclusion. The composite is considered to be composed of two confocal ellipses: an elliptical particle (σ -phase) surrounded by an elliptical region of continuous medium (β -phase). The far field flux outside of the outer ellipse is not disturbed by the presence of the particle. The governing equations and boundary conditions for this problem are given by

$$\nabla^2 T_\beta = 0 \quad \text{in } \beta\text{-phase} \quad (15)$$

$$\frac{\partial T_\beta}{\partial y} = 0 \quad \text{at } A_{s2} \quad (16)$$

$$\mathbf{n} \cdot k_\beta \nabla T_\beta = \mathbf{n} \cdot \mathbf{q}_\infty \quad \text{at } A_2 \quad (17)$$

$$\begin{cases} T_\beta = T_\sigma \\ \mathbf{n} \cdot k_\beta \nabla T_\beta = \mathbf{n} \cdot k_\sigma \nabla T_\sigma \end{cases} \quad \text{at } A_1 \quad (18)$$

$$\frac{\partial T_\sigma}{\partial y} = 0 \quad \text{at } A_{s1}, A_c \quad (19)$$

$$\nabla^2 T_\sigma = 0 \quad \text{in } \sigma\text{-phase} \quad (20)$$

To solve the boundary value problem described by Eq. (15) – (20) for the geometry in Fig. 3 elliptic cylinder coordinate transformation is used. The equations of transformation are given by

$$\begin{cases} x = c \cosh \eta \cos \psi \\ y = c \sinh \eta \sin \psi \end{cases} \quad (21)$$

Under the non-linear transformation of Eq. (21) the geometry of the two phase composite (Fig. 2) is converted to two rectangular domains as shown in Fig. 4.

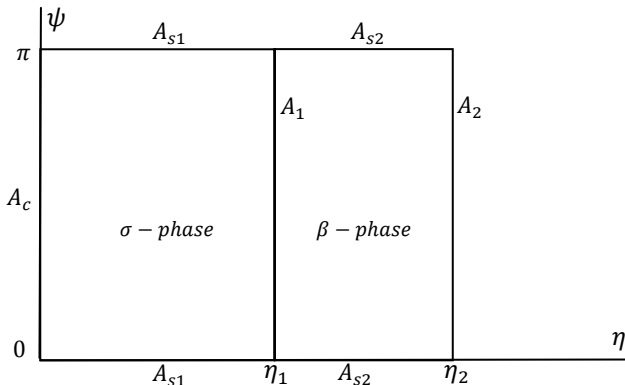


FIGURE 4. GEOMETRY FOR ETC IN X-DIRECTION AFTER TRANSFORMATION

The governing equations and boundary conditions in the new coordinate system take the following form.

$$\frac{\partial^2 T_\beta}{\partial \eta^2} + \frac{\partial^2 T_\beta}{\partial \psi^2} = 0 \quad \text{in } \beta\text{-phase} \quad (22)$$

$$\frac{\partial T_\beta}{\partial \psi} = 0 \quad \text{at } A_{s2} \quad (23)$$

$$\frac{\partial T_\beta}{\partial \eta} = \left(-\frac{\bar{q}_0^{-1} c}{k_\beta} \right) \sinh \eta_1 \cos \psi \quad \text{at } A_2 \quad (24)$$

$$\begin{cases} T_\beta = T_\sigma \\ k_\beta \frac{\partial T_\beta}{\partial \eta} = k_\sigma \frac{\partial T_\sigma}{\partial \eta} \end{cases} \quad \text{at } A_1 \quad (25)$$

$$\frac{\partial T_\sigma}{\partial \psi} = 0 \quad \text{at } A_{s1} \quad (26)$$

$$\frac{\partial T_\sigma}{\partial \eta} = 0 \quad \text{at } A_c \quad (27)$$

$$\frac{\partial^2 T_\sigma}{\partial \eta^2} + \frac{\partial^2 T_\sigma}{\partial \psi^2} = 0 \quad \text{in } \sigma\text{-phase} \quad (28)$$

where

$$\eta_1 = Ln \frac{w + \ell}{c}, \quad \eta_2 = Ln \frac{W + L}{c} \quad (29)$$

This boundary value problem has a straightforward solution for the geometry shown in Fig. 4. The solution can be written as

$$T_\beta(\eta, \psi) = \sum_{n=1}^{\infty} [B_n \cosh \lambda_n \eta + D_n \sinh \lambda_n \eta] \cos \lambda_n \psi \quad (30)$$

$$T_\sigma(\eta, \psi) = \sum_{n=1}^{\infty} A_n \cosh \lambda_n \eta \cos \lambda_n \psi \quad (31)$$

The coefficients A_n , B_n and D_n are found to be

$$A_n = A'_n P_n / Q_n \quad (32)$$

$$B_n = B'_n P_n / Q_n \quad (33)$$

$$D_n = D'_n P_n / Q_n \quad (34)$$

where

$$A'_n = \coth \lambda_n \eta_1 - \tanh \lambda_n \eta_1 \quad (35)$$

$$B'_n = \coth \lambda_n \eta_1 - \kappa \tanh \lambda_n \eta_1 \quad (36)$$

$$D'_n = \kappa - 1 \quad (37)$$

$$Q_n = \kappa [\coth \lambda_n \eta_2 - \tanh \lambda_n \eta_1] + [\coth \lambda_n \eta_1 - \coth \lambda_n \eta_2] \quad (38)$$

and

$$P_n = \begin{cases} \left(-\frac{\bar{q}_0^{-1} c}{k_\beta} \right) \frac{\sinh \eta_2}{\lambda_n \cosh \lambda_n \eta_2}, & n = 1 \\ 0, & n \neq 1 \end{cases} \quad (39)$$

In the above equations $\kappa = k_\beta / k_\sigma$ and $\lambda_n = n$. Equation (39) implies that only the first term in the infinite series solutions given by Eq. (30), (31) is nonzero.

Given that ETC tensor is diagonal for this case and using the definition given by Eq. (4), one can evaluate the ETC in x-direction as

$$K_{eff}^{xx} = \frac{\langle \frac{\partial T_\beta}{\partial x} \rangle + \kappa \langle \frac{\partial T_\sigma}{\partial x} \rangle}{\langle \frac{\partial T_\beta}{\partial x} \rangle + \langle \frac{\partial T_\sigma}{\partial x} \rangle} \quad (40)$$

Using the temperature distribution in Eq. (30) and (31), the ETC can be obtained from Eq. (40).

$$\frac{K_{eff}^{xx}}{k_\beta} = \frac{H_x^{(1)} + \kappa F_x^{(1)} - \varepsilon_\beta(\kappa - 1)}{H_x^{(2)} + \kappa F_x^{(2)} + \varepsilon_\beta(\kappa - 1)} \quad (41)$$

in which

$$H_x^{(1)} = \frac{(a_\beta - a_\sigma)(1 - a_\beta a_\sigma)}{a_\beta(1 - a_\sigma^2)} \quad (42)$$

$$H_x^{(2)} = \frac{(3a_\beta - a_\sigma) - a_\beta a_\sigma(a_\beta + a_\sigma)}{a_\beta(1 - a_\sigma^2)} \quad (43)$$

$$F_x^{(1)} = \frac{(a_\beta + a_\sigma) + a_\beta a_\sigma(a_\beta - 3a_\sigma)}{a_\beta(1 - a_\sigma^2)} \quad (44)$$

$$F_x^{(2)} = \frac{(a_\sigma - a_\beta)(1 - a_\beta a_\sigma)}{a_\beta(1 - a_\sigma^2)} \quad (45)$$

where the two aspect ratios are defined as

$$a_\beta = \frac{W}{L}, \quad a_\sigma = \frac{w}{\ell} \quad (46)$$

It is worth mentioning that the coefficients defined by Eq. (42) – (45) reflect the anisotropy of the composite in that they only depend on the aspect ratios. Therefore it is convenient to express them in matrix form.

$$\mathbf{M}_x = \begin{bmatrix} H_x^{(1)} & F_x^{(1)} \\ H_x^{(2)} & F_x^{(2)} \end{bmatrix} \quad (47)$$

where \mathbf{M}_x is defined as *anisotropy coefficients matrix* of the first principle component of ETC.

Effective thermal conductivity in y-direction

The procedure for obtaining the second principle component of ETC is the same as for the first one. In this case, the heat flow direction is considered to be along the minor diameter of ellipse. The geometry under consideration is presented in Fig. 5.

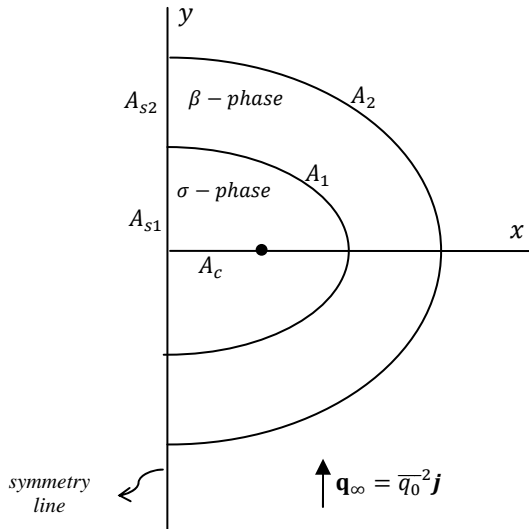


FIGURE 5. GEOMETRY FOR ETC IN Y-DIRECTION

All the dimensions in Fig. 5 are defined the same as for Fig. 3. The geometry after the transformation given by Eq. (21) is presented in Fig. 6.

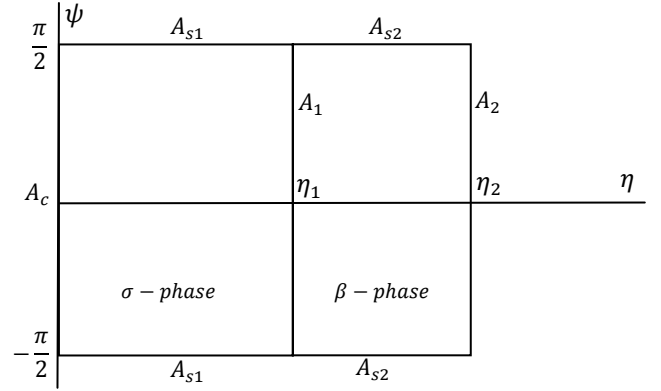


FIGURE 6. GEOMETRY FOR ETC IN Y-DIRECTION AFTER TRANSFORMATION

Following the same procedure as in the case of ETC in x-direction, the temperature distribution for β and σ -phases are obtained as

$$T_\beta(\eta, \psi) = \sum_{n=1}^{\infty} [B_n \cosh \lambda_n \eta + D_n \sinh \lambda_n \eta] \sin \lambda_n \psi \quad (48)$$

$$T_\sigma(\eta, \psi) = \sum_{n=1}^{\infty} A_n \sinh \lambda_n \eta \sin \lambda_n \psi \quad (49)$$

Using the definitions given by Eq. (32) – (34) for A_n , B_n and D_n we have

$$A'_n = \coth \lambda_n \eta_1 - \tanh \lambda_n \eta_1 \quad (50)$$

$$B'_n = 1 - \kappa \quad (51)$$

$$D'_n = \kappa \coth \lambda_n \eta_1 - \tanh \lambda_n \eta_1 \quad (52)$$

$$Q_n = \kappa [\coth \lambda_n \eta_1 - \tanh \lambda_n \eta_2] + [\tanh \lambda_n \eta_2 - \tanh \lambda_n \eta_1] \quad (53)$$

and

$$P_n = \begin{cases} \left(-\frac{\bar{q}_0^2 c}{k_\beta} \right) \frac{\cosh \eta_2}{\lambda_n \cosh \lambda_n \eta_2}, & n = 1 \\ 0, & n \neq 1 \end{cases} \quad (54)$$

where $\lambda_n = 2n - 1$. Effective thermal conductivity in y-direction can be obtained similar to the xx-component of ETC.

$$\frac{K_{eff}^{yy}}{k_\beta} = \frac{\langle \frac{\partial T_\beta}{\partial y} \rangle + \kappa \langle \frac{\partial T_\sigma}{\partial y} \rangle}{\langle \frac{\partial T_\beta}{\partial y} \rangle + \langle \frac{\partial T_\sigma}{\partial y} \rangle} \quad (55)$$

Equation (55) along with the temperature distributions of Eq. (48), (49) provides the following result

$$\frac{K_{eff}^{yy}}{k_\beta} = \frac{H_y^{(1)} + \kappa F_y^{(1)} - \varepsilon_\beta(\kappa - 1)}{H_y^{(2)} + \kappa F_y^{(2)} + \varepsilon_\beta(\kappa - 1)} \quad (56)$$

The anisotropy coefficients matrix of the second principle component of ETC is related to that of first principle component through the following relation

$$\mathbf{M}_y = \begin{bmatrix} H_y^{(1)} & F_y^{(1)} \\ H_y^{(2)} & F_y^{(2)} \end{bmatrix} = \begin{bmatrix} F_x^{(2)} & H_x^{(2)} \\ F_x^{(1)} & H_x^{(1)} \end{bmatrix} \quad (57)$$

RESULTS

Using the coordinate transformation equations introduced in the previous sections, one can obtain the complete ETC tensor with respect to an arbitrary rectangular coordinate system from its two principle components. The ETC of

composite shown in Fig. 2 with respect to $O_{\bar{X}^1\bar{X}^2}$ can be written as

$$\bar{\mathbf{K}}_{\text{eff}} = \begin{bmatrix} K_{\text{eff}}^{xx} & 0 \\ 0 & K_{\text{eff}}^{yy} \end{bmatrix} \quad (58)$$

where K_{eff}^{xx} and K_{eff}^{yy} are given by Eq. (41), (56) respectively. Employing (10) and (14), the following relation is obtained for ETC with respect to $O_{X^1X^2}$.

$$\mathbf{K}_{\text{eff}} = \begin{bmatrix} \cos^2 \theta K_{\text{eff}}^{xx} + \sin^2 \theta K_{\text{eff}}^{yy} & \sin \theta \cos \theta (K_{\text{eff}}^{xx} - K_{\text{eff}}^{yy}) \\ \sin \theta \cos \theta (K_{\text{eff}}^{xx} - K_{\text{eff}}^{yy}) & \sin^2 \theta K_{\text{eff}}^{xx} + \cos^2 \theta K_{\text{eff}}^{yy} \end{bmatrix} \quad (59)$$

Equation (59) describes the complete ETC tensor for a composite with random distributed elliptical inclusions and arbitrary orientation. It should also be mentioned that the presented model has led to a symmetric ETC tensor which was demonstrated in several references [1, 11, 13]. Furthermore, Eq. (59) is only valid for systems with dilute dispersion particles. Therefore, the exact shape of the outer ellipse (Fig. 3) has no significant effect on ETC and the anisotropy coefficients given by Eq. (42) – (45) can be simplified as

$$\mathbf{M}_{\mathbf{x}} = \lim_{a_\beta \rightarrow 1} \begin{bmatrix} H_x^{(1)} & F_x^{(1)} \\ H_x^{(2)} & F_x^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{1 - a_\sigma}{1 + a_\sigma} & \frac{1 + 3a_\sigma}{1 + a_\sigma} \\ \frac{3 + a_\sigma}{1 + a_\sigma} & \frac{a_\sigma - 1}{1 + a_\sigma} \end{bmatrix} \quad (60)$$

For circular cylindrical particles both aspect ratios in Eq. (46) approach one. In this limit the anisotropy coefficients matrices are given by

$$\lim_{a_\beta, a_\sigma \rightarrow 1} \mathbf{M}_{\mathbf{x}} = \lim_{a_\beta, a_\sigma \rightarrow 1} \mathbf{M}_{\mathbf{y}} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad (61)$$

which results in the following expression for xx and yy-component of ETC

$$K_{\text{eff}}^{xx} = K_{\text{eff}}^{yy} = \frac{2\kappa - \varepsilon_\beta(\kappa - 1)}{2 + \varepsilon_\beta(\kappa - 1)} \quad (62)$$

Substituting Eq. (62) in Eq. (59), the ETC tensor can be written as

$$\mathbf{K}_{\text{eff}} = K_{\text{eff}} \mathbf{I} \quad (63)$$

where $K_{\text{eff}} = K_{\text{eff}}^{xx} = K_{\text{eff}}^{yy}$. Equation (62) is actually Maxwell's model for ETC of circular cylindrical particles in a continuous medium [11]. Hence, in the limit when the particles are circular, the presented model produces isotropic ETC tensor with Maxwell's equation as its magnitude.

In general when $\kappa > 1$, decreasing the aspect ratio of particles increases the ETC along the major diameter and decreases it along the minor diameter. However, the situation for ETC along an arbitrary axis direction is not as straightforward. Figure 7 illustrates the effect of aspect ratio on the components of ETC for a particle rotated $\pi/6$ radian relative to X^1 -axis (see Fig. 2).

In obtaining Fig. 7, the simplification of Eq. (60) was used for the calculation of anisotropy coefficients matrices. It is observed that the diagonal components, i.e. K_{eff}^{11} and K_{eff}^{22} , approach the Maxwell's model and the cross components (K_{eff}^{12} and K_{eff}^{21}) approach zero when $a_\sigma \rightarrow 1$ as the inclusion becomes circular in this limit. It should also be mentioned that by decreasing the aspect ratio of inclusion, the cross components as well as diagonal components increase. This implies that when the major axis of particle is not along the coordinate system axes, heat conduction enhances in both X^1

and X^2 directions as the aspect ratio decreases. To clarify this point, the effect of angle between ellipse major axis and X^1 -axis, θ , is presented in Fig. 8.

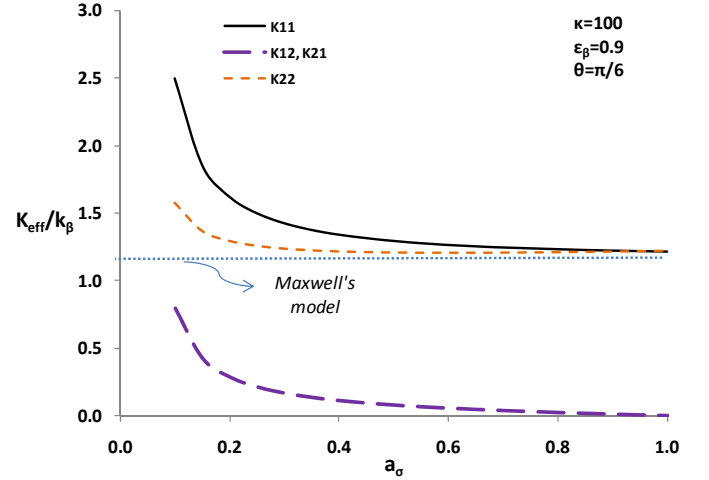


FIGURE 7. EFFECT OF ASPECT RATION ON ETC TENSOR, $\theta = \frac{\pi}{6}, \varepsilon_\beta = 0.9, \kappa = 100$

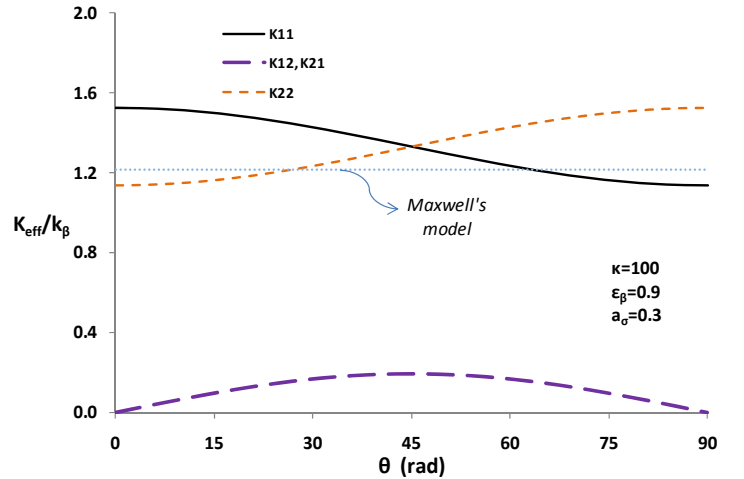


FIGURE 8. EFFECT OF PARTICLE ORIENTATION ON ETC TENSOR, $a_\sigma = 0.3, \varepsilon_\beta = 0.9, \kappa = 100$

As for Fig. 7, Eq. (60) is used for anisotropy coefficients matrices. Figure 8 shows the variation of all components of ETC versus particle orientation. The major axis of ellipse is along the X^1 and X^2 -axes for $\theta = 0$ and $\theta = \pi/2$, respectively. In comparison with circular cylindrical particles, the ETC along the major axis is higher and along the minor axis is lower, when $\kappa > 1$. Since the ellipse is stretched along the X^1 -axis when $\theta = 0$, K_{eff}^{11} is higher and K_{eff}^{22} is lower than Maxwell's model. For the same reason, K_{eff}^{11} is lower and K_{eff}^{22} is higher than Maxwell's model, when $\theta = \pi/2$. The cross components are zero in both cases. However, when θ is close to $\pi/4$, both diagonal components are higher than circular cylindrical particles and the cross coefficients are non-zero. This suggests that heat conduction in a low conductivity material in certain direction can be effectively increased by adding high conductivity non-circular inclusions stretched along that direction. Nevertheless, one can increase the heat conduction in two different directions at the same time by adding high conductivity non-circular inclusions oriented at an angle between the two directions.

SUMMARY AND CONCLUSIONS

Using the self-consistent field concept, an analytical solution for complete description of the effective thermal conductivity tensor of anisotropic composites was developed. It was shown that the presented solution yields the Maxwell's theory at the limit where the particles' aspect ratio approaches unity. The anisotropy of the structure was taken into account by introducing the anisotropy coefficients matrix into the Maxwell's equation which was only a function of particles aspect ratio. The effective thermal conductivity was expressed as a tensor associated with rectangular Cartesian transformation group that enables one to evaluate it along any arbitrary direction. It was found that the heat conduction in two different directions can be increased at the same time by adding high conductivity non-circular inclusions oriented at an angle between the two directions. The present solution can serve as a general model for anisotropic composites with dilute dispersion of particles.

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